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# MACROSCOPIC PROPERTIES OF A TWO-PHASE POTENTIAL DISPERSION COMPOSED OF IDENTICAL UNIT CELLS

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**Abstract**—The potential flow solution for flow of fluid past dispersed objects in a "unit cell" is used to derive several macroscopic properties, including the mean pressures in the phases and on the walls, the momentum and kinetic energy density, the force function and mechanical energy flux. These properties are derived from the "resistivity" of the unit cell, which has a tensorial character in general. Various macroscopic forms of Bernoulli's equation relate the properties. Equations of motion for uniform arrays of cells are derived. Various other features, such as minimization of kinetic energy density and forces at concentration jumps, are analyzed.

*Key Words:* two-phase, potential flow, averaging, macroscopic properties, Bernoulli, momentum flux, mechanical energy conservation

## 1. INTRODUCTION

Since the publication of books by Wallis (1969) and Ishii (1975), there has been a concerted attempt to develop a "two-fluid model" containing conservation equations to describe two-phase flows at a macroscopic level. Wallis (1969) asserted that this model could be useful for some flow regimes but cautioned about the appropriate evaluation of all terms (he himself made an error in formulating the added mass contributions!). Ishii (1975) provided a more general theoretical foundation in terms of *averages* but offered little help with the *unknown terms*, most of which represent interactions between the phases. Later authors made the leap to postulated *closure laws*, many of which can be shown to be inconsistent when applied to simple limiting cases that can be analyzed rigorously (Wallis 1992a).

Another approach (e.g. Batchelor 1970; Biesheuvel & van Wijngaarden 1984; Biesheuvel & Spoelstra 1989) has been to solve specific problems from a fundamental standpoint, usually involving some approximation but revealing relationships that might have a more general validity.

For didactic purposes there is a lack of illustrative examples of both methods that are sufficiently simple to be solved completely without invoking any hypotheses about closure. These provide instructive models for the rigorous derivation of macroscopic laws resembling their microscopic ancestors and containing no "missing" terms, perhaps with some clearly-stated idealizations. They serve to illustrate relationships, such as the interplay between Reynolds stresses in the fluid and internal stresses in dispersed particles, that may be harder to grasp and derive at a more sophisticated level. They also exemplify techniques, such as energy methods, and may reveal how apparently independent properties can be derived from each other. At the same time, it is clear what is "left out" and what idealizations are being made, so that extrapolation to more complex situations can be done with awareness. From the pedagogical viewpoint, this approach may be a useful precursor to more sophisticated presentations [e.g. section 3.4.1 of Drew & Wallis (1992) or section 4 of this paper] which, though more general, involve few conceptual advances and are harder to assimilate.

The purpose of this paper is to use basic methods of potential flow theory to derive the macroscopic properties of the fluid and the objects in *unit cells* that are sufficiently simple in structure to allow essentially exact mathematical solution. These unit cells are of interest in their own right, since they can form the basis for physical or numerical experiments. Moreover, they may also be used to represent a set of identical objects, or arrays of objects, arranged along the axis of a tube and in repetitive two- and three-dimensional arrays. To emphasize the equivalent treatment of these several systems, the analysis will be developed, as far as possible, in a way that applies to all of them. However, the one-dimensional cells are not merely simpler cases of three-dimensional ones but exhibit separate features, such as the average pressure on the wall and conservation of energy for each phase along the same "pipe" along which both the phases are constrained to flow (which is not the case in three-dimensions).

Such an approach is common in many branches of science; for example, for the determination of the effective electrical and mechanical properties of composite materials, an analogue applied to two-phase flow by Wallis (1989) and Smereka & Milton (1991). The method is to solve the basic equations at the microscopic level in the unit cell, using either analytical or numerical methods, and to perform suitable integration to determine macroscopic properties. In some cases, as in this paper, it transpires that theorems can be deduced that establish *constitutive equations*, or other fundamental laws that are obeyed by the macroscopic properties, often resembling the form of laws already known at the microscopic level. When properties can be interrelated, the key question then becomes, as in thermodynamics, whether some minimum set of independent properties suffices to provide a complete description. For the class of two-phase systems considered here, all the results can be expressed in terms of one dimensionless property, $\beta$, resembling a *resistivity* that is derivable by solving Laplace's equation. $\beta$ Depends on the geometrical structure of the two-phase interface and is generally a tensor.

Solutions to Laplace's equation are relatively easy to obtain for simple geometries, such as a row of equidistant spheres in a round tube (Cai & Wallis 1992), enabling several of the relationships between properties to be checked for specific cases. Such results have been obtained by Cai (1992) using numerical methods and found to be consistent with equations presented in this paper, particularly those describing the average pressure on the wall of a cell and within the dispersed phase. The present paper, however, is limited to the presentation of general theorems and their interpretation in conceptual terms which are highlighted by being written in italics in the ensuing text.

Figure 1 shows four examples of *unit cells*. Figure 1(a) represents one element of a long row of identical axisymmetrical equidistant incompressible objects located along the axis of a tube. In figure 1(b) there are many objects (particles or a porous medium) in the cell but, if the results here are to be valid, they must be constrained to move with the same velocity in the direction of the tube axis.

Independent objects which are not arranged symmetrically will experience non-uniform forces of interaction when there is any motion and the structure of the array will be subject to continual change. Moreover, when particles do not have uniform velocities, their "thermal" motion is an additional physical property which cannot be specified solely in terms of the average velocities of the phases. The way in which these "*thermal*" fluctuations in structure and velocity relate to the *average* motion is a separate major topic. This paper is concerned with the (limiting) *zero particle temperature* case.

To simplify conceptualization we start by assuming that the array is suitably symmetrical such that the fluid velocity is axial at the ends of the cell, though it is not a necessary restriction. At this stage we avoid consideration of how to treat particles that are cut by the cell boundaries by decreeing that there is a particle-free zone on all the faces of every cell. If necessary, this restriction can be lifted by invoking the characteristics of "cyclic" cells.

Figure 1(c) shows a rectangular, brick-like, unit cell of sides $X$, $Y$ and $Z$ ($X_i$ for short) with an object of three-fold symmetry at its center. We shall later consider more general objects with other shapes and locations. Figure 1(d) is a rectangular unit cell with contents resembling those in figure 1(b).

The fluid is inviscid, incompressible and irrotational and obeys the normal laws applicable to potential flow; in particular, the microscopic velocity is the gradient of a potential:

$$\mathbf{u} = -\nabla\phi. \qquad [1]$$

Though the application of potential flow theory to two-phase flow has all the limitations that apply to its use for single-phase flow, it nonetheless provides solutions for several limiting or ideal situations and serves as an example of how a self-consistent macroscopic theory may be developed.

In this paper we first analyze the simple case where the dispersed phase and cell are at rest, introduce the concept of *resistivity*, which was the basis of Wallis's (1989) analysis of isotropic dispersions, and relate it to the *polarization* of the particles or the *interphase impulse*. An extension is then made to describe moving cells in order to derive average *equations of motion* for each phase and expressions for the macroscopic *momentum and energy density*.

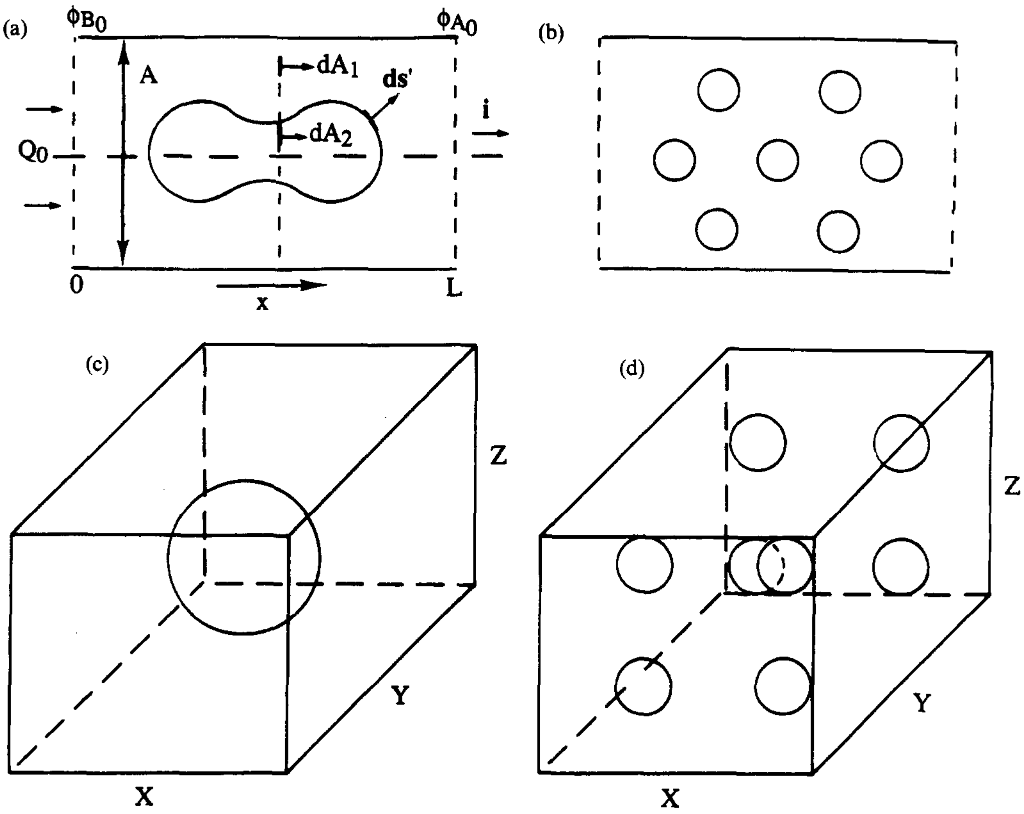

Figure 1. Various unit cells: (a,b) one-dimensional; (c,d) three-dimensional.

In section 3 further *volumetric averages* are taken in order to derive the *momentum flux*, *force function*, *mean pressures and stresses* within the phases and at *boundaries*, *energy fluxes* and *Bernoulli's equation*, i.e. several "macroscopic" properties that appear in mechanistic analysis and conservation laws. All of these results are strictly valid only for homogeneous arrays of cells and some mathematical justification must be given if they are to be applied to cells that change their shape and structure, for instance, in response to convective acceleration or concentration gradients. Some results, such as Bernoulli's equation and the related equation of overall energy conservation, apply to flows from one region to another, independent of the path. We note that it is customary to use the transport properties of composite media (and those derived from kinetic theory) even when there are gradients in properties and it would be interesting to investigate the order of approximation involved in following a similar treatment (e.g. Wallis 1991) of two-phase flows. In section 4 some analogous results are derived for three-dimensional cells. Though possibly useful for the development of more universal continuum models, these generalizations represent no conceptual advances and are more remote from direct numerical and experimental verification.

## 2. INTERPHASE IMPULSE, EQUATIONS OF MOTION, MOMENTUM AND ENERGY DENSITY

### 2.1. Base Case—Stationary Particles

Consider first the "base case" in which the dispersed phase is at rest. When it is necessary to make a distinction, velocities, potentials and fluxes in this reference frame will be assigned the subscript "0". The net flow of fluid is the same at any section perpendicular to the $x$-direction, which is the pipe axis for the one-dimensional flow patterns:

$$Q_0 = \int \mathbf{u} \cdot \mathrm{d}A_1 \mathbf{i}. \qquad [2]$$

Integrating [2] over the cell from $x = 0$ to $L$,

$$Q_0 L = \int \mathbf{u} \cdot \mathrm{d}A_1 \mathbf{i} \, \mathrm{d}L = \int u_x \, \mathrm{d}V_1. \qquad [3]$$

Dividing by the total volume of the cell, $V = AL$, we get

$$j_0 = \frac{Q_0}{A} = \frac{\int u_x \, \mathrm{d}V_1}{V} = \frac{V_1}{V} \frac{\int u_x \, \mathrm{d}V_1}{V_1} = \epsilon_1 w, \qquad [4]$$

where $\epsilon_1$ is the fraction of the volume occupied by the fluid and $w$ is the average fluid velocity in the $x$-direction, or the "relative" velocity; $j_0$ is the fluid flux in this reference frame.

For the three-dimensional cells, [4] can also be expressed in vector form as

$$\mathbf{j}_0 = \epsilon_1 \mathbf{w} = \frac{\int \mathbf{u} \, \mathrm{d}V_1}{V}. \qquad [5]$$

The total fluid momentum in the $x$-direction in a one-dimensional unit may be expressed as

$$\rho_1 \int \mathbf{u} \, \mathrm{d}V_1 = -\rho_1 \int \nabla \phi \, \mathrm{d}V_1 = -\rho_1 \int \phi \, \mathbf{ds} = -\rho_1 \left[ (\phi_{A_0} - \phi_{B_0}) A \mathbf{i} - \int \phi_0 \, \mathbf{ds}' \right], \qquad [6]$$

where $\phi_{A_0}$ and $\phi_{B_0}$ are the potentials at the ends of the cell, which are uniform because of symmetry, $\mathbf{ds}$ is an element of the surface pointing outward from the fluid and $\mathbf{ds}'$ is an element of area on the surface of the object, pointing out into the fluid. For the three-dimensional cells the equation has the same form but with three components to the term involving potential difference.

We define $\beta$ (Wallis 1989) as the factor by which the potential difference across the unit cell is increased by reason of the presence of the object in the base case:

$$\phi_{B_0} - \phi_{A_0} = \beta L j_0. \qquad [7]$$

$\beta$ Resembles a dimensionless *resistivity* dependent on the geometry of the cell and can be determined by solving Laplace's equation.

The three-dimensional case may not be isotropic and $\beta$ will have components $\beta_i$ in the three directions, therefore

$$\phi_{B_{0i}} - \phi_{A_{0i}} = \beta_i X_i j_{0i}. \qquad [8]$$

As a result of symmetry, we consider the axes of the cell to be *principal axes* so that coefficients linking fluxes and flows in different directions do not appear; $\beta_i$ are the principal components of a *tensor resistivity*, $\boldsymbol{\beta}$. Then, from [4], [6] and [7], for the one-dimensional cells,

$$\int \phi_0 \, \mathbf{ds}' = \left( \frac{1}{\beta} - 1 \right) A (\phi_{B_0} - \phi_{A_0}) \mathbf{i} = (1 - \beta) \mathbf{w} \epsilon_1 V, \qquad [9]$$

while the equivalent of [9] for the three-dimensional cells is

$$\frac{\int \phi_0 \, \mathbf{ds}'}{V} = (\mathbf{I} - \boldsymbol{\beta}) \cdot \mathbf{w} \epsilon_1, \qquad [10]$$

where $\mathbf{I}$ is the unit tensor (identity matrix).

The quantity in [9] is the same as the net dipole moment of internal sources that could have been used to "generate" the dispersed phase (Wallis 1991, 1992b) and could be called the *polarization*. Equations [9] and [10] indicate that, for a given structure of the cell, this *polarization* is proportional to the relative velocity and described by a *constitutive relation* (Wallis 1992b). Moreover, the left-hand sides of [9] and [10] are measures of an *interphase impulse* necessary to set up the motion from the rest. This mutual impulse is an example of an *interaction* term in the two-phase flow. If $\beta$ can be determined by solving Laplace's equations, [9] and [10] are *constitutive equations* relating the mutual impulse to the relative motion.

### 2.2. *Moving Particles—Equations of Motion*

We now set the particles in motion in any of the cells in figure 1 by superposing an additional potential field,

$$\phi_2 = -\mathbf{v}_2 \cdot \mathbf{r}, \qquad [11]$$

which acts only in the $x$-direction in the one-dimensional case. The average fluid velocity is then

$$\mathbf{v}_1 = \mathbf{w} + \mathbf{v}_2 \qquad [12]$$

and the potential everywhere is

$$\phi = \phi_0 + \phi_2. \qquad [13]$$

Integrating [11] over the particle(s) in a cell

$$\int \phi_2 \, \mathbf{ds}' = -\int \mathbf{v}_2 \cdot \mathbf{r} \mathbf{ds}' = -\mathbf{v}_2 \epsilon_2 V, \qquad [14]$$

where $\epsilon_2$ is the fraction of the volume occupied by the particles.

Adding [14] to [9] and using [12] and [13] we obtain, when both phases are in motion,

$$\frac{\int \phi \, \mathbf{ds}'}{V} = (1-\beta)\epsilon_1 \mathbf{w} - \epsilon_2 \mathbf{v}_2 = (1-\epsilon_1\beta)\mathbf{w} - \epsilon_2 \mathbf{v}_1 \qquad [15]$$

for the one-dimensional cells and, more generally, for the three-dimensional case:

$$\frac{\int \phi \, \mathbf{ds}'}{V} = (\mathbf{I} - \epsilon_1 \boldsymbol{\beta}) \cdot \mathbf{w} - \epsilon_2 \mathbf{v}_1. \qquad [16]$$

Equations [15] and [16] will be useful for evaluating one of the key terms in the net force, due to surface pressure, acting on the dispersed phase (hereafter called the "object") during a transient in which both phases accelerate. For a start we consider the base case when the object is at rest and the flow is steady. Since pressures and velocities are identical at corresponding points at both ends of the unit cells, a momentum balance shows that there is no net force on the object (barring body force fields which can be added later). Therefore, the external force is applied to it is

$$\mathbf{F}_2 = \int p \, \mathbf{ds}' = \int (p_0 - \tfrac{1}{2}\rho_1 u_0^2) \mathbf{ds}' = -\frac{\rho_1}{2} \int u_0^2 \mathbf{ds}' = 0, \qquad [17]$$

where Bernoulli's equation was used with $p_0$ being the stagnation pressure.

Consider now a more general condition where there is a gravitational field "$g$" and both phases are accelerating. The equation of motion of the object is

$$\rho_2 V_2 \dot{\mathbf{v}}_2 = \mathbf{F}_2 + \rho_2 \mathbf{g} V_2 - \int p \, \mathbf{ds}', \qquad [18]$$

where $p$ may again be derived from Bernoulli's equation as

$$p = p_0 - \tfrac{1}{2}\rho_1 u^2 + \rho_1 \frac{\partial \phi}{\partial t} + \mathbf{g} \cdot (\mathbf{r} - \mathbf{r}_0)\rho_1. \qquad [19]$$

The stagnation pressure, $p_0$, may be a function of time, but at any instant its integral, $\int p_0 \mathbf{ds}'$, is zero. The velocity is related to the base case velocity by

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}_2 \qquad [20]$$

and the rate of change of $\phi$ in the lab system, $\partial\phi/\partial t$, is related to the rate of change of $\phi$ in a system moving with the particles, $\mathrm{d}\phi/\mathrm{d}t$, by

$$\frac{\partial \phi}{\partial t} = \frac{\mathrm{d}\phi}{\mathrm{d}t} - \mathbf{v}_2 \cdot \nabla \phi_0 = \frac{\mathrm{d}\phi}{\mathrm{d}t} + \mathbf{v}_2 \cdot \mathbf{u}_0. \qquad [21]$$

Using [20] and [21] in terms in [19] in which those expressions appear, we have

$$-\tfrac{1}{2}\rho u^2 + \rho_1 \frac{\partial \phi}{\partial t} = -\tfrac{1}{2}\rho_1 u_0^2 - \tfrac{1}{2}\rho v_2^2 + \rho_1 \frac{\mathrm{d}\phi}{\mathrm{d}t}. \qquad [22]$$

Substituting [22] into [19], integrating over the particle surface and using [15] and [17], [18] becomes

$$\rho_2 V_2 \dot{\mathbf{v}}_2 = \mathbf{F}_2 + \rho_2 \mathbf{g} V_2 + \rho_1 V[(\epsilon_1 \beta - 1)\dot{\mathbf{w}} + \epsilon_2 \dot{\mathbf{v}}_1)] - \rho_1 \mathbf{g} V_2. \qquad [23]$$

All the constant terms in [20] vanished when integrated over the surface of the object.

Dividing [23] by $V_2$ and defining the external force per unit volume of phase 2 as

$$\mathbf{f}_2 = \frac{\mathbf{F}_2}{V_2}, \qquad [24]$$

we obtain the *equation of motion of the dispersed phase* in the form

$$\rho_2 \dot{\mathbf{v}}_2 - \rho_1 \dot{\mathbf{v}}_1 - \rho_1 \frac{\epsilon_1 \beta - 1}{\epsilon_2} \dot{\mathbf{w}} = \mathbf{f}_2 + \rho_2 \mathbf{g} - \rho_1 \mathbf{g}. \qquad [25]$$

The role of the *interphase impulse*, from [15], in the case where such a motion is set up suddenly, is evident in the second and third terms on the left-hand side.

The corresponding result for the three-dimensional case is obtained by using [16] instead of [15] and is

$$\rho_2 \dot{\mathbf{v}}_2 - \rho_1 \dot{\mathbf{v}}_1 - \rho_1 \frac{(\epsilon_1 \boldsymbol{\beta} - \mathbf{I})}{\epsilon_2} \cdot \dot{\mathbf{w}} = \mathbf{f}_2 + \rho_2 \mathbf{g} - \rho_1 \mathbf{g}. \qquad [26]$$

Now, the overall momentum equation for the entire volume of fluid plus the particle in figure 1 is

$$\rho_1 V_1 \dot{\mathbf{v}}_1 + \rho_2 V_2 \dot{\mathbf{v}}_2 = \mathbf{f}_2 V_2 + \rho_1 \mathbf{g} V_1 + \rho_2 \mathbf{g} V_2 - (A_x \Delta p_x \mathbf{i} + A_y \Delta p_y \mathbf{j} + A_z \Delta p_z \mathbf{k}), \qquad [27]$$

where $\Delta p$ is the pressure difference between (any) corresponding points at the ends of the control volume and only the *i*-direction is relevant for the one-dimensional systems. Dividing [27] by the total volume $V$ and defining the *macroscopic pressure gradient* as

$$\nabla P = \frac{\Delta p_x}{X} \mathbf{i} + \frac{\Delta p_y}{Y} \mathbf{j} + \frac{\Delta p_z}{Z} \mathbf{k}, \qquad [28]$$

[27] may be written as

$$\rho_1 \epsilon_1 \dot{\mathbf{v}}_1 + \rho_2 \epsilon_2 \dot{\mathbf{v}}_2 = \epsilon_2 \mathbf{f}_2 + (\rho_1 \epsilon_1 + \rho_2 \epsilon_2) \mathbf{g} - \nabla P. \qquad [29]$$

Equations [25] and [29] may now be combined to give further results which are convenient. Adding $\epsilon_1 \times$ [25] to [29] gives

$$\rho_2 \dot{\mathbf{v}}_2 - \frac{\epsilon_1}{\epsilon_2} \rho_1 (\epsilon_1 \beta - 1) \dot{\mathbf{w}} = \mathbf{f}_2 + \rho_2 \mathbf{g} - \nabla P, \qquad [30]$$

whereas subtracting $\epsilon_2 \times$ [25] from [29] gives

$$\rho_1 \dot{\mathbf{v}}_1 + \rho_1 (\epsilon_1 \beta - 1) \dot{\mathbf{w}} = \rho_1 \mathbf{g} - \nabla P. \qquad [31]$$

Equations [30] and [31] were derived by Wallis (1989) using a different approach (energy conservation) and interpreted as *equations of motion for each phase* with the term involving $\dot{\mathbf{w}}$ describing *inertial coupling* or *added mass* effects. While purists might regard [25] as being more truly the "equation of motion of the particles", [30] and [31] have the virtue of symmetry and are more directly related to *two-fluid models* of two-phase flow. The combination $(\epsilon_1 \beta - 1)$ is a geometrical parameter that can, in principle, be derived by solving Laplace's equation and has been called the *exertia* (Wallis 1989), since it describes the "external inertia" associated with the dispersed phase when it moves relative to the fluid.

The equivalents of [30] and [31] for the three-dimensional case are

$$\rho_2 \dot{\mathbf{v}}_2 - \frac{\epsilon_1}{\epsilon_2} \rho_1 (\epsilon_1 \boldsymbol{\beta} - \mathbf{I}) \cdot \dot{\mathbf{w}} = \mathbf{f}_2 + \rho_2 \mathbf{g} - \nabla P \qquad [32]$$

and

$$\rho_1 \dot{\mathbf{v}}_1 + \rho_1 (\epsilon_1 \boldsymbol{\beta} - \mathbf{I}) \cdot \dot{\mathbf{w}} = \rho_1 \mathbf{g} - \nabla P. \qquad [33]$$

Inertial coupling in this situation has a tensor nature, as does the classical *added mass* for a non-rotating single particle. If the cell is isotropic, [32] and [33] collapse to [30] and [31] which are then valid in both the one- and three-dimensional cases.

While many authors have written down equations of the form of [30]–[33], there has been remarkable confusion about how the inertial coupling term is to be evaluated (e.g. the use of Zuber's added mass coefficient to replace $\epsilon_1\beta - 1$). An unequivocal definition has been given here in terms of quantities that are directly derivable from solutions to Laplace's equation.

### 2.3. Coefficients of Added Mass

Equations [15], [30] and [31], or the equivalent three-dimensional versions, may be used to define a "coefficient of added mass". For example, if there is no mean pressure gradient or gravitational force, [30] and [31] may be combined to give

$$\dot{\mathbf{v}}_2\left[\rho_2 + \rho_1\left(\frac{\epsilon_1\beta - 1}{\epsilon_2\beta}\right)\right] = \mathbf{f}_2. \qquad [34]$$

In another case an external force $\mathbf{f}_2$ might be applied and the constraint imposed that there is zero net flux, as in a closed-end container:

$$\epsilon_1\dot{\mathbf{v}}_1 + \epsilon_2\dot{\mathbf{v}}_2 = 0. \qquad [35]$$

Then [26] becomes

$$\dot{\mathbf{v}}_2\left[\rho_2 + \rho_1\left(\frac{\beta - 1}{\epsilon_2} - 1\right)\right] = \mathbf{f}_2 + \rho_2 g - \rho_1 g. \qquad [36]$$

The quantities in parentheses in [34] and [36] might be *defined* as *coefficients of added mass* for those particular circumstances, but they are not the same quantity. In fact, the *added mass coefficient* depends on the boundary conditions imposed on the unit cell, specifically the relationship between the potential difference across it and the net *flux*, or *bulk velocity*, through it, which can be expressed as an *external impedance*. In [34] there is no *external impedance*, whereas in [36] the *external impedance* is infinite. A whole range of added mass coefficients may be obtained by connecting a series of unit cells to an external fluid loop, much as a battery with internal resistance may be connected to an external circuit. The external loop specifies a third relationship between $\nabla P$ and $\epsilon_1\dot{\mathbf{v}}_1 + \epsilon_2\dot{\mathbf{v}}_2$ which is to be solved simultaneously with [30] and [31] to determine the relationship of $\mathbf{f}_2$ to $\dot{\mathbf{v}}_2$. Cai (1992) presents an analysis and experimental verification of these concepts.

In yet another case, the only external force applied might be a macroscopic pressure gradient, in which case we obtain from [30] and [31]

$$\dot{\mathbf{v}}_2\left[\rho_2 + (\rho_1 - \rho_2)\frac{\epsilon_1\beta - 1}{\beta - 1}\right] = -\nabla P, \qquad [37]$$

and no clear definition of an *added mass coefficient* emerges. The "added mass" requires careful interpretation and clear definition and is best derived from the more basic equations of motion.

### 2.4. Momentum and Energy Density

The quantities appearing in [15] and [16] are the *interphase impulse* associated with the motion of particles. For motion in an infinite fluid this impulse is directly related to the overall momentum and kinetic energy of the fluid (Lamb 1945). A similar approach may be taken in the present case, except that it is now necessary to include integrals around the boundaries of the unit cell.

The net fluid momentum in a cell with both phases moving is

$$\rho_1\int \mathbf{u}\,\mathrm{d}V_1 = -\rho_1\int \nabla\phi\,\mathrm{d}V_1 = -\rho_1\int \phi\,\mathbf{ds}, \qquad [38]$$

where the integral is taken over all of the boundaries and $\mathbf{ds}$ is an element of the boundary pointing outward from the fluid. Dividing [38] by the cell volume and splitting the surface integral into parts on the outer boundary of the cell and on the enclosed particles, the momentum density is

$$\mathbf{m}_1 = \rho_1\frac{\int \mathbf{u}\,\mathrm{d}V_1}{V} = \rho_1(\phi_B - \phi_A)\frac{A}{V}\mathbf{i} + \rho_1\frac{\int \phi\,\mathbf{ds}'}{V}. \qquad [39]$$

Using [7], [11] and [13] in the first term and [15] in the second we obtain, for the one-dimensional cells,

$$\mathbf{m}_1 = \rho_1(\epsilon_1\beta\mathbf{w} + \mathbf{v}_2) - \rho_1[(\epsilon_1\beta - 1)\mathbf{w} + \epsilon_2\mathbf{v}_1] \tag{40}$$

which reduces to

$$\mathbf{m}_1 = \rho_1\epsilon_1\mathbf{v}_1 \tag{41}$$

as expected.

The corresponding results for the three-dimensional cells are

$$m_{1i} = \rho_1(\phi_{\mathrm{B}_i} - \phi_{\mathrm{A}_i})\frac{A_i}{V} + \rho_1\int\frac{\phi\,\mathrm{d}s'_i}{V} \tag{42}$$

which becomes, using [8], [11], [13] and [16],

$$m_{1i} = \rho_1(\epsilon_1\beta_i w_i + v_{2i}) + \rho_1[(w_i - \epsilon_1\beta_i w_i) - \epsilon_2 v_{1i}] \tag{43}$$

or, on simplification, as expected,

$$m_{1i} = \rho_1\epsilon_1 v_{1i}. \tag{44}$$

The kinetic energy density follows in a similar way from the basic relation:

$$k_1 = \frac{\frac{1}{2}\rho_1\displaystyle\int u^2\,\mathrm{d}V_1}{V} = -\frac{\rho_1}{2V}\int\phi\mathbf{u}\cdot\mathbf{ds}. \tag{45}$$

The part of the integral around the particles is transformed by using the boundary condition at the particle surface:

$$(\mathbf{u} - \mathbf{v}_2)\cdot\mathbf{ds} = 0 \tag{46}$$

so that, for the particles,

$$\int\phi\mathbf{u}\cdot\mathbf{ds} = \int\phi\mathbf{v}_2\cdot\mathbf{ds} = -\mathbf{v}_2\cdot\int\phi\mathbf{ds}'. \tag{47}$$

Evaluating [45] for the entire fluid surface, including the cell boundary and the particles, we have, for the one-dimensional cells,

$$k_1 = \tfrac{1}{2}\rho_1\frac{(\phi_{\mathrm{B}} - \phi_{\mathrm{A}})}{L}\mathbf{i}\cdot(\epsilon_1\mathbf{v}_1 + \epsilon_2\mathbf{v}_2) - \frac{\rho_1}{2}\mathbf{v}_2\cdot[(\epsilon_1\beta - 1)\mathbf{w} + \epsilon_2\mathbf{v}_1] \tag{48}$$

$$= \tfrac{1}{2}\rho_1[(\epsilon_1\beta\mathbf{w} + \mathbf{v}_2)\cdot(\epsilon_1\mathbf{v}_1 + \epsilon_2\mathbf{v}_2) - \mathbf{v}_2\cdot\mathbf{w}(\epsilon_1 - 1) - \mathbf{v}_2\cdot\mathbf{v}_1\epsilon_2] \tag{49}$$

$$= \tfrac{1}{2}\rho_1\epsilon_1 v_1^2 + \tfrac{1}{2}\rho_1\epsilon_1(\epsilon_1\beta - 1)w^2 \tag{50}$$

which is consistent with Wallis (1989).

The corresponding result for the three-dimensional cells is

$$k_1 = \frac{\rho_1}{2}\sum_i\left[(\phi_{\mathrm{B}_i} - \phi_{\mathrm{A}_i})A_i j_i + v_{2i}\int\phi\,\mathrm{d}s'_i\right] \tag{51}$$

$$= \frac{\rho_1}{2}\sum_i[(\epsilon_1\beta_i w_i + v_{2i})(\epsilon_1 v_{1i} + \epsilon_2 v_{2i}) - v_{2i}w_i(\epsilon_1\beta_i - 1) - v_{2i}v_{1i}\epsilon_2] \tag{52}$$

$$= \tfrac{1}{2}\rho_1\epsilon_1 v_1^2 + \tfrac{1}{2}\rho_1\epsilon_1\sum_i(\epsilon_1\beta_i - 1)w_i^2. \tag{53}$$

A considerable advantage of the above methods over some in use in the literature is that there is no need to "go to infinity" since integrals are only needed around surfaces in the unit cell. Possible problems of convergence are thus avoided.

Some alternative models (e.g. Zuber 1964) are based on a unit cell with *fixed* boundaries inside which the dispersed phase is free to move. This removes the integral around the cell boundary from [45], but not from [38], and enables the kinetic energy to be determined, leading to a definition of the "added mass". However, this constraint is artificial for the motion of a large array since the cells move with the particles, except perhaps near boundaries. Nevertheless, the two approaches may lead to estimates of some macroscopic properties, such as added mass coefficients when there is no net flux, that are (approximately) compatible (Wallis 1989). In general, the problem of a unit cell with impermeable boundaries is *not* the same as the corresponding problem with equipotential boundaries and the two solutions cannot be derived from each other. Each situation is physically realizable, but under different constraints. There is also a further class of problems involving small numbers of cells (e.g. four equidistant spheres on the axis of a closed-end tube) that are intermediate between the two limiting cases and are best treated on an individual basis if accuracy is desired.

### 2.5. *Non-symmetrical Particles, More General Cases*

If the particles are unsymmetrical or uncentered in the unit cell (i.e. they have any shape and are anywhere), things are a little more complicated. There are, in general, no planes "A" and "B" across which the potentials are equal. However, because of the cyclic nature of the unit cells, potential differences across corresponding points at opposite ends of the cell are equal and the quantity $(\phi_B - \phi_A)$ is still uniform and unequivocal. In the case of the particles in a tube, only the component of [17] along the axis is zero and there may be transverse forces on the particles from the fluid. Equations [5]–[31] are still valid as far as one-dimensional motion in the direction of the tube axis is concerned, however, as long as the particles are constrained to remain in the same position in the unit cell. A similar unspecified (mutual) force in the direction perpendicular to the relative motion appeared in Wallis's (1989) analysis that was based on energy conservation.

The cyclic nature of the three-dimensional rectangular array assures that in the base case the velocities (and hence from Bernoulli's equation also the pressures) repeat at corresponding points in the unit cells. Therefore, there should be no hydrodynamic force acting on the particles in the base case. The analysis leading to [32] and [33] is equally valid for this situation. Similar "cyclic" arguments may be used to include cases where particles are cut by the borders of control volumes. While theorems, similar to those presented here, may be derived for such general cases as a simple generalization, the actual derivation of properties may be complicated by uncertainty in specifying unequivocal boundary conditions (unless more conditions, such as planes of symmetry, can be imposed).

The general case, when there are no unit cells and the particles are both randomly distributed and randomly oriented, has to be treated in some statistical way that is beyond the scope of the present analysis. It would be interesting to see if the form of [32] and [33] can be obtained for this situation. However, severe new complications arise because interactions between particles cause continual rearrangement of the structure of the array. Moreover, *ensemble averaging* approaches should strictly consider all possible interactions and positions of all particles, which is hard to accomplish except in the *dilute limit.*

In all of the above analysis the unit cells were all identical and there were no gradients of the macroscopic variables, such as $\epsilon_1$, $v_1$ and $v_2$. In the case of convective accelerations the unit cell itself distorts and $\beta$ changes in response to the flow field, producing an interesting challenge to analysis. There is a possibility that the structure of the particle array might depend on the entire previous flow history, which would be awkward for making practical predictions. The case where the structure "relaxes" to some standard form and $\beta$ depends only on the volumetric concentration, $\epsilon_2$, is discussed by Wallis (1991).

## 3. MOMENTUM FLUX, MEAN STRESSES AND PRESSURES, BERNOULLI'S EQUATION AND ENERGY CONSERVATION FOR ONE-DIMENSIONAL UNIT CELLS

In this section it will be shown how several *macroscopic properties* can be derived once the *resistivity*, $\beta$, is derived as a function of the geometry of the cell. The properties are first derived with particles at rest and then with particles in motion.

### 3.1. *Base Case*

Since no external forces are applied, the sum of the rate of momentum transport and the internal forces across any cross-section perpendicular to the axis of the cell must be constant with a value denoted by $F$, resembling the "impulse function" defined by Shapiro (1953) and sometimes called the "*flow force*":

$$F = \int_{A_1} (p + \rho_1 u_{0x}^2)\,\mathrm{d}A_1 - \int_{A_2} \sigma_x\,\mathrm{d}A_2; \qquad [54]$$

$\sigma_x$ is the normal tensile stress in the $x$-direction within the particle. Integrating [54] over the length of the unit cell we obtain

$$FL = \int (p + \rho_1 u_{0x}^2)\,\mathrm{d}V_1 - \int \sigma_x\,\mathrm{d}V_2 \qquad [55]$$

or, dividing by the total cell volume $V = AL$,

$$\frac{F}{A} = \epsilon_1 p_1 + \rho_1 \epsilon_1 \langle u_{0x}^2 \rangle - \epsilon_2 \langle \sigma_x \rangle, \qquad [56]$$

where the symbol $\langle\ \rangle$ denotes the volume average over a particular phase and $p_1 = \langle p \rangle$ for the fluid. The momentum average may alternatively be expressed as

$$\langle u_{0x}^2 \rangle = w^2 + \langle \tilde{u}_{0x}^2 \rangle, \qquad [57]$$

where $w$ is the average velocity of the fluid and $\tilde{u}_{0x}$ is a fluctuating component that gives rise to "Reynolds stresses".

An alternative approach is to use control volumes which go around the particles and contain only fluid. The one shown in figure 2 extends from $x = 0$, where properties are described by the subscript "$b$", to $x = x$. Areas in the flow are perpendicular to the $x$-direction with elements $\mathrm{d}A_1$, while elements of areas on the object are denoted by $\mathbf{ds}'$ pointing out from the surface. Since pressures on the sides cancel in a symmetrical unit cell, the momentum balance for this volume is

$$\int_b [p_b \mathbf{i} + \rho u_{bx}(u_{bx}\mathbf{i} + u_{by}\mathbf{j} + u_{bz}\mathbf{k})]\,\mathrm{d}A_1 + \int_0^x p\,\mathbf{ds}' - \int_x [p\mathbf{i} + \rho u_x(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})]\,\mathrm{d}A_1 = 0, \qquad [58]$$

which is valid for all $x$. We integrate it from 0 to $L$ and divide by the total volume, $V = AL$.

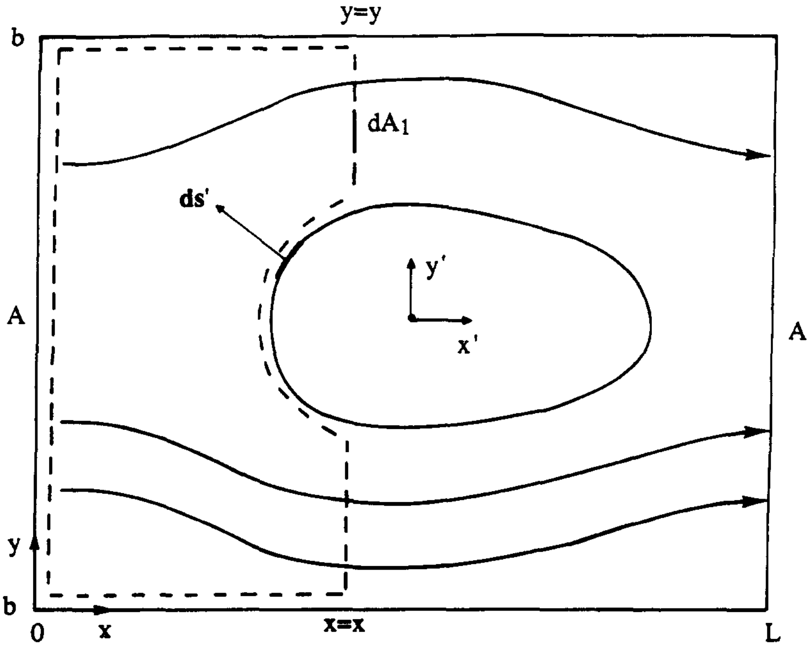

Figure 2. Control volume, with $x$-direction faces in the fluid, passing around the dispersed phase.

The integral involving pressure is evaluated "by parts":

$$\int_0^L \int_0^x p\mathbf{ds}'\,\mathrm{d}x = \left[x\int_0^x p\mathbf{ds}'\right]_0^L - \int_0^L xp\mathbf{ds}'. \qquad [59]$$

The first term on the right is zero because $\int_0^L p\mathbf{ds}'$ is the net force on the object and already deduced to be zero from an overall momentum balance.

The resulting three components of the "averaged" equation [58] are:

$$\frac{\int xp\,\mathrm{d}s_x'}{V} + \epsilon_1 p_1 + \rho(w^2 + \langle \tilde{u}_{0x}^2\rangle)\epsilon_1 = \bar{p}_\mathrm{b} + \rho(\overline{u_{0x}^2})_\mathrm{b} = \frac{F}{A}, \qquad [60]$$

$$\frac{\int xp\,\mathrm{d}s_y'}{V} + \rho\langle \tilde{u}_{0x}\tilde{u}_{0y}\rangle\epsilon_1 = \rho(\overline{u_{0x}u_{0y}})_\mathrm{b} \qquad [61]$$

and

$$\frac{\int xp\,\mathrm{d}s_z'}{V} + \rho\langle \tilde{u}_{0x}\tilde{u}_{0z}\rangle\epsilon_1 = \rho(\overline{u_{0x}u_{0z}})_\mathrm{b}, \qquad [62]$$

where the "bar" denotes an average over the cross-sectional area and the tildes denote fluctuations from the volumetric average.

The three integrals appearing in [60]–[62] are products of the particle volume fraction and components of the *particle stress* or *interfacial stress tensor* $\int x_i p\,\mathrm{d}s_j'/V$. For the symmetrical centrally-located object that we have assumed, all of the terms in [61] and [62] vanish. In a more general situation the equations express a relationship between the interfacial stress tensor and the Reynolds stress tensor.

Comparing [60] with [56] confirms the identity derived by Batchelor (1970) and Wallis (1989), relating the average stress in the particle to the pressure on its surface:

$$-\langle\sigma_x\rangle = \int \frac{xp\,\mathrm{d}s_x'}{V_2}. \qquad [63]$$

Bernoulli's equation (we neglect body forces in the rest of this paper) is valid for any point in the fluid:

$$p + \tfrac{1}{2}\rho_1 u_1^2 = p_0, \qquad [64]$$

where $p_0$ is the stagnation pressure of the fluid. Making use of the theorem

$$\int u_0^2\,\mathrm{d}V_1 = -\int \phi\mathbf{u}\cdot\mathbf{ds} = (\phi_{\mathrm{B}_0} - \phi_{\mathrm{A}_0})j_0 A = \beta j_0^2 V = \beta\epsilon_1 w^2 V_1 \qquad [65]$$

we may integrate [64] over all of the fluid and divide by the fluid volume to obtain (Wallis 1989):

$$p_1 + \tfrac{1}{2}\rho_1 w^2\epsilon_1\beta = p_0, \qquad [66]$$

which is a special case of the *averaged Bernoulli equation for the fluid.*

Now, the flow force $F$ is not independent but can be derived from the kinetic energy in the cell by using the approach used by Wallis (1989). We consider a long tube with length $l \gg L$ of which a fraction "$f$" contains unit cells of the type shown in figure 1(a), while the remaining $(1-f)$ contains pure fluid (figure 3). The net potential drop across the tube, $\Delta\phi$, is related to the fluid volumetric flow rate, $Q_0 = Aj_0$, by

$$\Delta\phi = l[1 - f + f\beta]j_0 = BQ_0, \qquad [67]$$

where $B$ plays the role of a net "resistance". The total kinetic energy of the fluid is, using the same approach as in [65],

$$K = \tfrac{1}{2}\rho_1\,\Delta\phi Q_0 = \tfrac{1}{2}\rho_1\frac{\Delta\phi^2}{B}. \qquad [68]$$

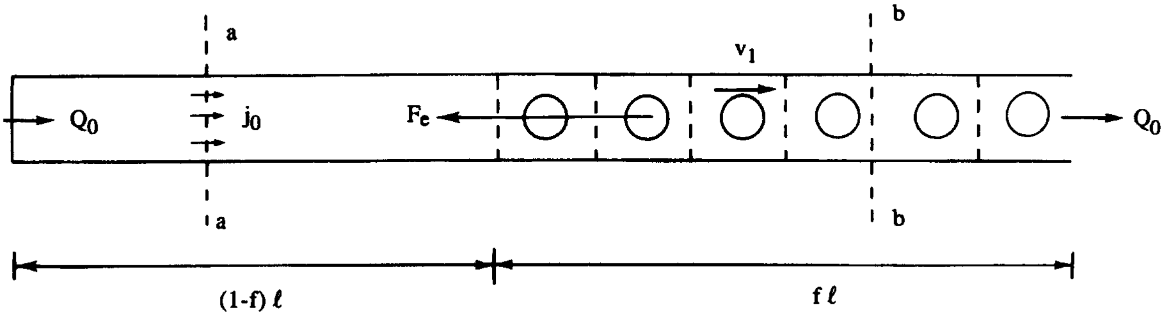

Figure 3. A long tube with a fraction "$f$" occupied by unit cells containing particles.

From Wallis (1989), the work done by moving any of the fluid–solid boundaries in figure 3, while maintaining $\Delta\phi$ fixed, equals the resulting change in fluid kinetic energy, i.e.

$$\mathrm{d}K = -\tfrac{1}{2}\rho_l \frac{\Delta\phi^2}{B^2}\,\mathrm{d}B \qquad [69]$$

or, from [67],

$$\mathrm{d}K = -\tfrac{1}{2}\rho_l Q_0^2\,\mathrm{d}B. \qquad [70]$$

We now consider a perturbation which all of the unit cells in figure 3 increase in length by an amount $\mathrm{d}L$, giving rise to a corresponding change in $f$, where

$$\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}L}{L}; \qquad [71]$$

it being understood that the extension $\mathrm{d}L$ occurs in a way that preserves the cell's symmetry.

Since this changes the net kinetic energy, a force must have done work. Now, the only external force that acts in figure 3 is the one that restrains particles near the interface with the pure fluid and extends over a few unit cells. If $l$ is sufficiently long, this force, $F_e$, may be regarded as concentrated at the interface between the pure fluid and the region containing particles, and we have

$$F_e l\,\mathrm{d}f = \mathrm{d}K = -\tfrac{1}{2}\rho_l Q_0^2\,\mathrm{d}B; \qquad [72]$$

$\mathrm{d}B$ may be evaluated from [67] and the result is

$$F_e = -\tfrac{1}{2}\rho_l \frac{Q_0^2}{A}\left(\beta + f\frac{\partial\beta}{\partial f} - 1\right). \qquad [73]$$

In view of [71] and [4], this may be written as

$$F_e = -\tfrac{1}{2}\rho_l j_0^2 A\left(\beta + L\frac{\partial\beta}{\partial L} - 1\right) \qquad [74]$$

which resembles [3.113] in Wallis (1989), except that $\beta$ is not a function of fluid volumetric fraction, $\epsilon_l$, alone but depends on all the dimensions which define the unit cell, such as its length and diameter as well as the dimensions and arrangement of the dispersed particles. $F_e$ is an external force which must be applied to keep the array in figure 3 from collapsing. It is related to the *particle pressure* (Wallis 1989) and the "effective" pressure mentioned by Geurst (1991), which play a role in determining the stability of the flow, since a particle "gas" must have a positive compressibility.

The momentum balance between sections "a" and "b" in figure 3 is

$$A(p_a + \rho_l v_a^2) - F_e = F. \qquad [75]$$

Now, the region "$a$" may be chosen far enough from the interface for fluid conditions to be uniform, therefore

$$v_a = j_0, \qquad [76]$$

and Bernoulli's equation may be used to give

$$p_a = p_0 - \tfrac{1}{2}\rho_1 j_0^2. \qquad [77]$$

Substituting [66], [74], [76] and [77] into [75], we obtain

$$\frac{F}{A} = p_1 + \tfrac{1}{2}\rho_1 j_0^2\left(\beta + \frac{\beta}{\epsilon_1} + L\frac{\partial\beta}{\partial L}\right) \qquad [78]$$

which can be combined with [56] to give

$$\rho_1\epsilon_1\langle u_{0x}^2\rangle - \epsilon_2\langle\sigma_x\rangle = \epsilon_2 p_1 + \tfrac{1}{2}\rho_1 j_0^2\left(\beta + \frac{\beta}{\epsilon_1} + L\frac{\partial\beta}{\partial L}\right). \qquad [79]$$

If $\beta$ is determined, as a function of the geometry of the unit cell, the term in parentheses in [78] and [79] can be evaluated and therefore the *flow force*, which is important in overall momentum balance, is known. The combination of the *Reynolds stresses* and the *particle stress* is also known, but not the two quantities independently.

Similar results for the stresses in the other directions are most easily obtained if the unit cell is rectangular, with overall dimensions $Y$ and $Z$ in the $y$- and $z$-directions. The flow force in these directions reduces to the mean wall pressure times the wall area and the equivalents of [56] are

$$\bar{p}_{wy} = \epsilon_1 p_1 + \rho_1\epsilon_1\langle u_{0y}^2\rangle - \epsilon_2\langle\sigma_y\rangle \qquad [80]$$

and

$$\bar{p}_{wz} = \epsilon_1 p_1 + \rho_1\epsilon_1\langle u_{0z}^2\rangle - \epsilon_2\langle\sigma_z\rangle. \qquad [81]$$

The equivalent of [58] for the $y$-direction follows from the control volume, sketched in figure 4, extending from $y = 0$ to $y = y$ and passing around the dispersed phase:

$$\int p_w\,\mathrm{d}A_{wy}\mathbf{j} + \int_0^y p\,\mathbf{ds}' - \int [p\mathbf{j} + \rho u_y(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})]\,\mathrm{d}A_y = 0. \qquad [82]$$

The momentum fluxes and pressures at the ends $x = 0$ and $x = L$ cancel. Integrating [82] from $y = 0$ to $y = Y$ and dividing by $V$ we get three components:

$$\frac{\int yp\,\mathrm{d}s'_x}{V} + \rho\langle\tilde{u}_{0y}\tilde{u}_{0x}\rangle\epsilon_1 = 0, \qquad [83]$$

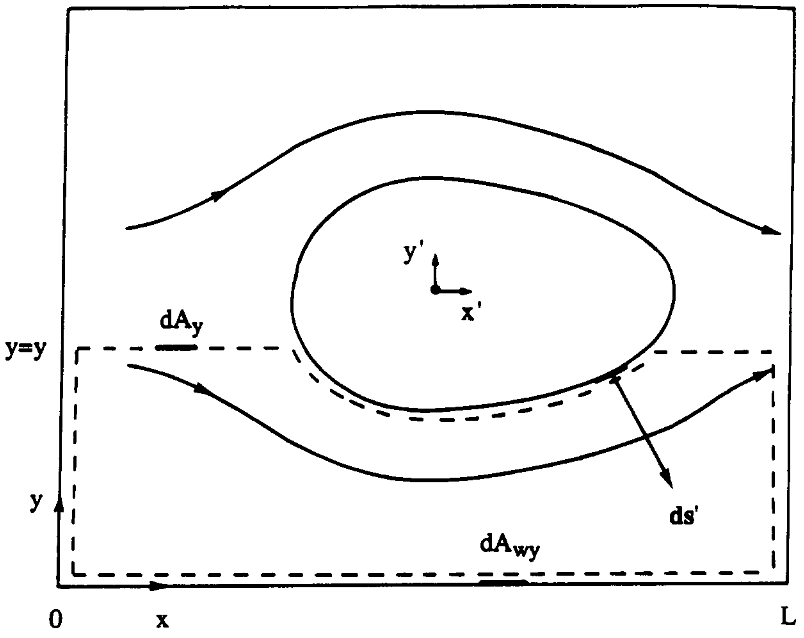

Figure 4. Control volume, with $y$-direction faces in the fluid, passing around the dispersed phase.

$$\epsilon_1 p_1 + \frac{\int yp \, \mathrm{d}s'_y}{V} + \rho \langle \tilde{u}_{0y} \tilde{u}_{0y} \rangle \epsilon_1 = \bar{p}_{wy} \qquad [84]$$

and

$$\frac{\int yp \, \mathrm{d}s'_z}{V} + \rho \langle \tilde{u}_{0y} \tilde{u}_{0z} \rangle \epsilon_1 = 0. \qquad [85]$$

Again, the integrals that result are components of the *interfacial stress tensor* ($\int x_i p \, \mathrm{d}s'_j / V$) and all of the terms in [83] and [85] are zero for a symmetrical centrally-located object—in other words, the axes of the cell are principal axes of this tensor.

Comparing [84] with [80] we obtain a result resembling [63], with $y$ replacing $x$. Similar expressions are valid for the $z$-direction.

Adding [79], [80] and [81] leads to

$$\bar{p}_{wy} + \bar{p}_{wz} = 2\epsilon_1 p_1 + p_1 \epsilon_1 (\langle u_{0x} \rangle^2 + \langle u_{0y} \rangle^2 + \langle u_{0z} \rangle^2)$$
$$- \epsilon_2 (\langle \sigma_x \rangle + \langle \sigma_y \rangle + \langle \sigma_z \rangle) - \epsilon_2 p_1 - \tfrac{1}{2} \rho_1 j_0^2 \left( \beta + \frac{\beta}{\epsilon_1} + L \frac{\partial \beta}{\partial L} \right). \qquad [86]$$

Now, from [65],

$$\langle u_{0x}^2 \rangle + \langle u_{0y}^2 \rangle + \langle u_{0z}^2 \rangle = \beta \epsilon_1 w^2 \qquad [87]$$

and the term involving particle stresses can be written

$$\langle \sigma_x \rangle + \langle \sigma_y \rangle + \langle \sigma_z \rangle = -3p_2, \qquad [88]$$

where $p_2$ is the mean bulk stress in the particle. Therefore, [86] can be rewritten as

$$\bar{p}_{wy} + \bar{p}_{wz} = 2(\epsilon_1 p_1 + \epsilon_2 p_2) + \epsilon_2 \left[ p_2 - p_1 - \tfrac{1}{2} \rho_1 j_0^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) \right], \qquad [89]$$

which is the special case of the equation for the *mean wall pressure*.

The average pressure in the dispersed phase may also be computed using an energy method (Wallis 1989). The boundaries of the unit cell are kept fixed, as well as the applied potential difference, $\phi_B - \phi_A$, while the particle undergoes a uniform volumetric strain (figure 5). If $\mathbf{r}$ is the vector from a fixed origin (normally the centroid of the particle, otherwise the strain involves displacement as well as change in volume; however, if there is no net force on the particle, no work is done in a displacement and any origin of strain may be chosen) to a point on the surface of the particle, this uniform strain is described by

$$\mathbf{dr} = \gamma \mathbf{r}, \qquad [90]$$

where $\gamma$ is constant over the particle. The change in particle volume at each point on its surface is

$$\mathrm{d}V_2 = \mathbf{dr} \cdot \mathbf{ds}' = \gamma \mathbf{r} \cdot \mathbf{ds}' \qquad [91]$$

and the net change in volume is

$$\mathrm{d}V_2 = \int \mathrm{d}V_2 = \gamma \int \mathbf{r} \cdot \mathbf{ds}' = 3\gamma V_2. \qquad [92]$$

The work done by the interfacial pressure during this expansion is

$$\int p \, \mathrm{d}V_2 = \int \gamma p \mathbf{r} \cdot \mathbf{ds}' = \gamma \int p(x \, \mathrm{d}s'_x + y \, \mathrm{d}s'_y + z \, \mathrm{d}s'_z). \qquad [93]$$

The terms on the right-hand side of [93] are three components that can be replaced by mean stresses in the particle, as in [63]. Using [88] and [92], we then have

$$\int p \, \mathrm{d}V_2 = 3\gamma p_2 V_2 = p_2 \, \mathrm{d}V_2. \qquad [94]$$

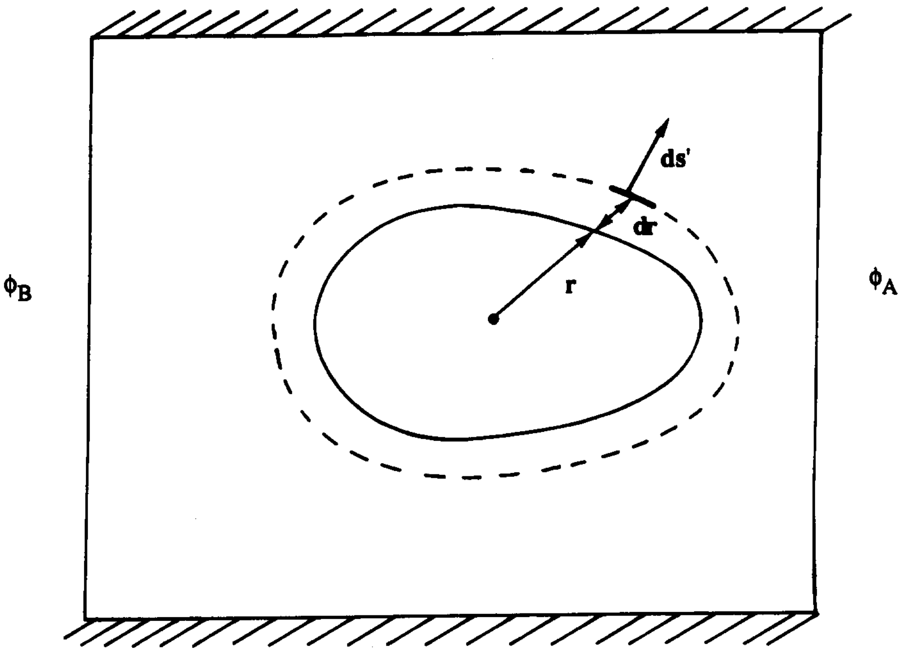

Figure 5. Particle(s) undergoing uniform strain.

In this case

$$B = \beta \frac{L}{A} \qquad [95]$$

and the equivalent of [70] is

$$\mathrm{d}K = -\tfrac{1}{2}\rho_1 Q_0^2 \frac{L}{A}\,\mathrm{d}\beta = -\tfrac{1}{2}\rho_1 j_0^2 V\,\mathrm{d}\beta \qquad [96]$$

and, since

$$\mathrm{d}V_2 = V\,\mathrm{d}\epsilon_2, \qquad [97]$$

we obtain

$$p_2 = p_0 - \tfrac{1}{2}\rho_1 j_0^2 \left(\frac{\partial \beta}{\partial \epsilon_2}\right)_{L,A}. \qquad [98]$$

The differential in [98] is evaluated by keeping the overall dimensions of the cell fixed and imposing a uniform strain on the particle; it provides a direct method of evaluating the *mean pressure in the particle* from knowledge of $\beta$.

The *mean wall pressure* may also be evaluated by a similar energy method in which the wall undergoes a uniform strain (figure 6). If $\mathrm{d}A_w$ is an element of area of the wall, pointing outwards from the fluid, the work done by the wall on the fluid during the strain is

$$W_w = -\int \gamma p \mathbf{r} \cdot \mathbf{dA}_w. \qquad [99]$$

The net volume change of the cell is

$$\mathrm{d}V = \mathbf{dr} \cdot \mathbf{dA}_w = \gamma \mathbf{r} \cdot \mathbf{dA}_w = 2\gamma V \qquad [100]$$

and the change of its area of cross section is

$$\mathrm{d}A = 2\gamma A. \qquad [101]$$

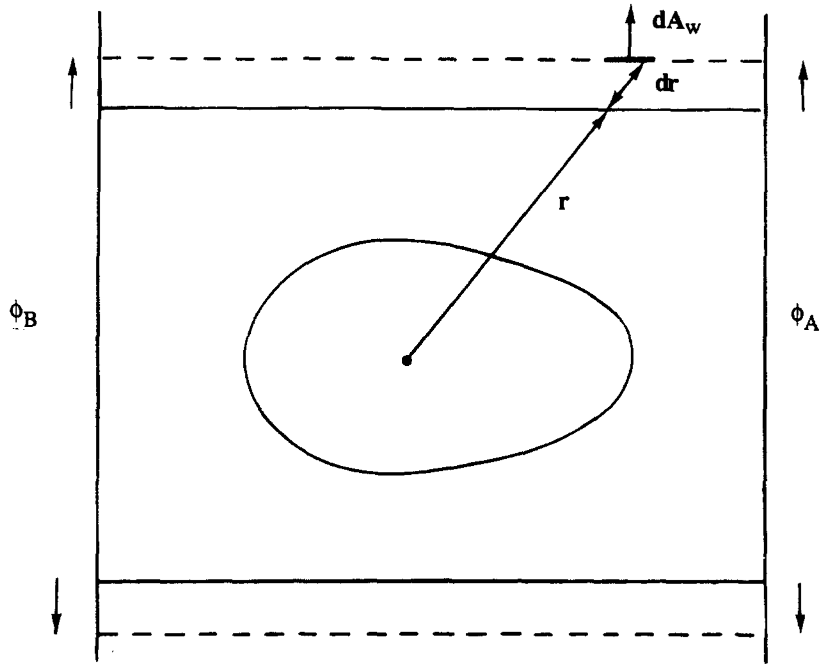

Figure 6. The wall of the unit cell undergoing uniform strain.

We define the *average pressure on the wall* of the cell as

$$p_w = \frac{\int p\mathbf{r}\cdot \mathbf{dA}_w}{\int \mathbf{r}\cdot \mathbf{dA}_w} = \frac{\int p\mathbf{r}\cdot \mathbf{dA}_w}{2V}, \qquad [102]$$

which in the case of the rectangular brick-like cell becomes

$$p_w = \frac{\bar{p}_{wy} + \bar{p}_{wz}}{2}. \qquad [103]$$

Then [99] can be expressed as

$$W_w = -p_w\,\mathrm{d}V. \qquad [104]$$

The change in resistance of the cell is evaluated from [95], noting that both $\beta$ and $A$ change

$$\mathrm{d}B = \frac{L}{A}\,\mathrm{d}\beta - \frac{L}{A^2}\beta\,\mathrm{d}A. \qquad [105]$$

The energy theorem for this case becomes

$$(p_0 - p_w)\mathrm{d}V = \mathrm{d}K = -\tfrac{1}{2}\rho_1 Q_0^2\left(\frac{L}{A}\,\mathrm{d}\beta - \frac{L}{A}\beta\frac{\mathrm{d}V}{V}\right), \qquad [106]$$

where [100] and [101] were used. Rearranging and introducing the fluid flux, $j_0$, we finally obtain

$$p_w = p_0 + \tfrac{1}{2}\rho_1 j_0^2\left(A\left(\frac{\partial\beta}{\partial A}\right)_{L,V_2} - \beta\right). \qquad [107]$$

The differential is to be performed observing the constraints indicated and provides a direct method of computing the *mean wall pressure* when $\beta$ is known. Equation [107] has been verified by Cai (1992) for several examples of exact solutions to the sphere-in-tube case [figure 1(a)].

Now, the three partial derivatives $\partial\beta/\partial\epsilon_2$, $\partial\beta/\partial A$ and $\partial\beta/\partial L$ are not independent. In a uniform strain, with its origin at the center of the cell, affecting *all* dimensions of the cell and the phase interface, the shape of the flow field is conserved and therefore $\beta$ is constant. In general, we have

$$\mathrm{d}\beta = \frac{\partial\beta}{\partial L}\,\mathrm{d}L + \frac{\partial\beta}{\partial\epsilon_2}\,\mathrm{d}\epsilon_2 + \frac{\partial\beta}{\partial A}\,\mathrm{d}A, \qquad [108]$$

while for global uniform strain

$$\gamma = \frac{\mathrm{d}L}{L} = \frac{1}{2}\frac{\mathrm{d}A}{A} = \frac{1}{3}\frac{\mathrm{d}\epsilon_2}{\epsilon_2}. \tag{109}$$

Since this does not change $\beta$, [108] becomes

$$O = L\frac{\partial \beta}{\partial L} + 2A\frac{\partial \beta}{\partial A} + 3\epsilon_2\frac{\partial \beta}{\partial \epsilon_2}. \tag{110}$$

Using [98] and [107] in [110], we get

$$\tfrac{1}{2}\rho_1 j_0^2\left(2\beta + L\frac{\partial \beta}{\partial L}\right) + 3\epsilon_2(p_0 - p_2) + 2(p_w - p_0) = 0. \tag{111}$$

We now form $(2\epsilon_1 - \epsilon_2) \times$ [66]:

$$(2\epsilon_1 - \epsilon_2)\left(p_0 - p_1 - \tfrac{1}{2}\rho_1 j_0^2\frac{\beta}{\epsilon_1}\right) = 0. \tag{112}$$

The sum of [111] and [112] is

$$p_w = (\epsilon_1 p_1 + \epsilon_2 p_2) + \frac{\epsilon_2}{2}\left[p_2 - p_1 - \tfrac{1}{2}\rho_1 j_0^2\left(\frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2}\frac{\partial \beta}{\partial L}\right)\right], \tag{113}$$

which is a more general version of [89] and shows that the *mean wall pressure* differs, in general, from the *volumetric mean pressure* in the dispersion unless the term in square brackets vanishes. The condition for this to occur is presented later in [121] and [122].

## *3.2. Moving Cell*

### *3.2.1. Flow force*

We now consider the pattern of unit cells to move along the pipe at speed $v_2$, which is also the speed of the particles. The time for a slice of the cell of thickness $\mathrm{d}x$ to pass a stationary observer is

$$\mathrm{d}t = \frac{\mathrm{d}x}{v_2}, \tag{114}$$

while the entire cell goes by in time

$$T = \frac{L}{v_2}. \tag{115}$$

The time average of the flow force at a fixed plane is

$$\langle F \rangle = \frac{1}{T}\int[(p + \rho_1 u_x^2)\,\mathrm{d}A_1 + (p_2 v_2^2 - \sigma_x)\,\mathrm{d}A_2]\mathrm{d}t, \tag{116}$$

which may be transformed, using [114] and [115] and relating velocities to the base case, into

$$\langle F \rangle = \frac{1}{L}\int\{[p + \rho_1(v_2 + u_{0x})^2]\,\mathrm{d}V_1 + (p_2 v_2^2 - \sigma_x)\,\mathrm{d}V_2\}. \tag{117}$$

Since

$$v_1 = v_2 + \langle u_{0x} \rangle = v_2 + w, \tag{118}$$

[117] may be expressed as

$$\langle F \rangle = A[\epsilon_1 p_1 + \epsilon_1 \rho_1 v_1^2 + \epsilon_2 \rho_2 v_2^2 + \epsilon_1 \rho_1(\langle u_{0x}^2 \rangle - w^2) - \epsilon_2\langle \sigma_x \rangle], \tag{119}$$

which contains no surprises, resembling the left-hand side of [3.148] in Wallis (1989). To make comparisons with the right-hand side of the latter equation we make use of [79] and find that the *flow force* may be expressed in terms of average properties and $\beta$ as

$$\frac{\langle F \rangle}{A} = \epsilon_1 p_1 + \epsilon_1 \rho_1 v_1^2 + \epsilon_2 \rho_2 v_2^2 + \epsilon_1 \rho_1 w^2(\epsilon_1 \beta - 1) + \epsilon_2\left[p_1 + \tfrac{1}{2}\rho_1 j_0^2\left(\frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2}\frac{\partial \beta}{\partial L}\right)\right]. \tag{120}$$

The term in square brackets in [120] simplifies in the *special* case where $\beta$ depends only on $\epsilon_2$, as was assumed by Wallis (1989) and should be true for any dispersion that maintains an *isotropic* random structure. From the geometry of the cell we then have

$$\frac{dL}{L}=-\frac{d\epsilon_2}{\epsilon_2}=\frac{d\epsilon_1}{\epsilon_2} \qquad [121]$$

and, therefore,

$$p_1+\tfrac{1}{2}\rho_1 j_0^2\left(\frac{\beta}{\epsilon_1}+\frac{L}{\epsilon_2}\frac{\partial\beta}{\partial L}\right)=p_1+\tfrac{1}{2}\rho_1 j_0^2\left(\frac{\beta}{\epsilon_1}+\frac{\partial\beta}{\partial\epsilon_1}\right) \qquad [122]$$

which equals $p_2$ for such a system (Wallis 1989). In this case the final term in [113] vanishes and the mean wall pressure is equal to the volumetric mean pressure in the two-phase mixture. The general problem of computing the *mean wall pressure* for any array is not so simple, however, since the structure will itself change during the virtual displacement described by [99].

### *3.2.2. Bernoulli's equation for the fluid*

Bernoulli's equation at any point in the fluid in the moving unit cell is

$$p_0=p+\tfrac{1}{2}\rho_1 u^2-\rho_1\frac{\partial\phi}{\partial t}, \qquad [123]$$

where $\phi$ is the microscopic velocity potential; $p_0$ is a constant throughout the flow and may be a function of time. Since a "reference" potential, $\phi_r$, can always be added everywhere in the flow, both sides of [123] are arbitrary by the amount $\rho_1\,\partial\phi_r/\partial t$ at any time and we may always choose this reference so that $\partial\phi/\partial t$ is set to zero at a particular "stagnation point", where $u$ also vanishes. However, $p_0$ is not the pressure at all stagnation points because there may be an additional contribution from $\partial\phi/\partial t$ there.

When the unit cell moves with velocity, $\mathbf{v}_2$, $\partial\phi/\partial t$ is non-zero as a result of the potential distribution in the base case moving past an observer in the laboratory frame; this contributes a portion

$$-\mathbf{v}_2\cdot\nabla\phi_0=\mathbf{v}_2\cdot\mathbf{u}_0=v_2 u_{0x}. \qquad [124]$$

However, if we simply move the original potential distribution past the observer, the potential will rise by the amount $\phi_B-\phi_A$ with the passage of each unit cell and this will create a rate of change of the macroscopic potential $\Phi$. Introducing the reference potential $\phi_r$, we then have

$$\frac{\partial\Phi}{\partial t}=v_2\frac{\phi_B-\phi_A}{L}-\frac{\partial\phi_r}{\partial t} \qquad [125]$$

while, from [124],

$$\frac{\partial\phi}{\partial t}=v_2 u_{0x}-\frac{\partial\phi_r}{\partial t}. \qquad [126]$$

Combining [125] and [126] to eliminate $\phi_r$ we obtain

$$\frac{\partial\phi}{\partial t}=\frac{\partial\Phi}{\partial t}+v_2\left(u_{0x}-\frac{\phi_B-\phi_A}{L}\right)=\frac{\partial\Phi}{\partial t}+v_2(u_{0x}-\beta\epsilon_1 w). \qquad [127]$$

Equation [127] is then used in [123], which may be integrated across slices of fluid passing a fixed location and over the time of passage of a unit cell, as in [116]. This is then transformed, using [114] and [115], in the same way that [117] followed from [116], to give

$$p_1=\frac{1}{V_1}\int\left[p_0-\tfrac{1}{2}\rho_1(\mathbf{v}_2+\mathbf{u}_0)^2+\rho_1 v_2(u_{0x}-\beta\epsilon_1 w)+\rho_1\frac{\partial\Phi}{\partial t}\right]dV_1. \qquad [128]$$

With the help of [65] and [124] this becomes

$$p_1=p_0-\tfrac{1}{2}\rho_1 v_2^2-\tfrac{1}{2}\rho_1\epsilon_1\beta w^2-\rho_1\epsilon_1\beta v_2 w+\rho_1\frac{\partial\Phi}{\partial t}, \qquad [129]$$

which may be expressed, using [118], as

$$p_1 = p_0 - \tfrac{1}{2}\rho_1 v_1^2 - \tfrac{1}{2}\rho_1(\epsilon_1\beta - 1)(v_1^2 - v_2^2) + \rho_1 \frac{\partial\Phi}{\partial t}. \qquad [130]$$

Equation [130] is the *averaged Bernoulli equation* for the fluid (Wallis 1989, 1991). It interrelates the macroscopic properties of *any* unit cells, not necessarily of the same structure or orientation, as long as they are connected by a continuous path through the fluid. The idea of averaging Bernoulli's equation was used earlier by Voinov & Petrov (1977) who ignored terms involving the square of velocity perturbations due to the particle. However, these terms are of the same order as the other components of kinetic energy (Wallis 1991) and must be retained.

When the flow is "macroscopically steady", the term $\partial\Phi/\partial t$ may be disgarded. However, it requires careful evaluation when there are unsteady features in the flow field, even though these are far away from regions that may appear to contain flows that are locally steady. This point may be illustrated by the example sketched in figure 7.

In figure 7(a) the conditions are macroscopically steady, with flow from a single-phase region "I" to a region "II" where particles are at rest, the mean fluid velocity is $v_1$ and the volume fraction is $\epsilon_1$. In figure 7(b) a velocity $v_2$ has been imposed on the entire flow. Because of the movement of the interface between regions I and II, the flow is macroscopically unsteady, though it appears locally steady in each region.

In the steady flow case the fluid velocity, from continuity, is $\epsilon_1 v_1$ in region I and the mean fluid pressure follows from [130]

$$p_{1\mathrm{I}} = p_0 - \tfrac{1}{2}\rho_1 \epsilon_1^2 v_1^2. \qquad [131]$$

In region II the corresponding result is

$$p_{1\mathrm{II}} = p_0 - \tfrac{1}{2}\rho_1 \epsilon_1 \beta v_1^2. \qquad [132]$$

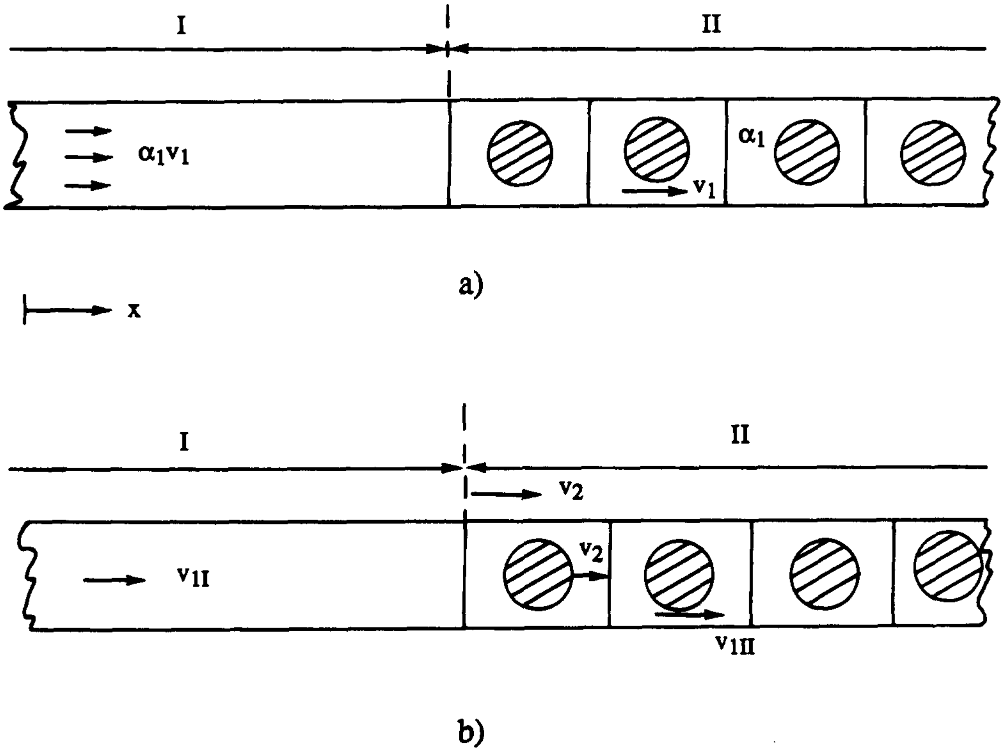

Figure 7. Region I—pure fluid; region II—unit cells. (a) Stationary particles and cells; (b) particles and cells with speed $v_2$.

Subtracting [132] from [131] we obtain

$$p_{\rm II} - p_{\rm III} = \tfrac{1}{2}\rho_1\epsilon_1 v_1^2(\beta - \epsilon_1). \qquad [133]$$

The macroscopic potential gradients in regions I and II in figure 7(a) are

$$-\frac{\mathrm{d}\Phi_{\rm I}}{\mathrm{d}x} = \epsilon_1 v_1 \qquad [134]$$

and

$$-\frac{\mathrm{d}\Phi_{\rm II}}{\mathrm{d}x} = \beta\epsilon_1 v_1. \qquad [135]$$

The mean fluid velocities in figure 7(b) are related to those in figure 7(a) by

$$v_{\rm II} = \epsilon_1 v_1 + v_2 \qquad [136]$$

and

$$v_{\rm III} = v_1 + v_2. \qquad [137]$$

Using these values in [130], the mean pressures in the two regions are found to be

$$p_{\rm II} = p_0 - \tfrac{1}{2}\rho_1(\epsilon_1 v_1 + v_2)^2 + \rho_1\frac{\partial\Phi_{\rm I}}{\partial t} \qquad [138]$$

and

$$p_{\rm III} = p_0 - \tfrac{1}{2}\rho_1(v_1 + v_2)^2 + \tfrac{1}{2}\rho_1(\epsilon_1\beta - 1)[(v_1 + v_2)^2 - v_2^2] + \rho_1\frac{\partial\Phi_{\rm II}}{\partial t}. \qquad [139]$$

Now, by use of a suitable reference potential, either of the unsteady terms in [138] or [139] may be removed and the flow regarded as "steady" in that region. However, both of these terms cannot be removed simultaneously because, from [134] and [135],

$$\frac{\partial\Phi_{\rm I}}{\partial t} - \frac{\partial\Phi_{\rm II}}{\partial t} = -v_2\left(\frac{\mathrm{d}\Phi_{\rm I}}{\mathrm{d}x} - \frac{\mathrm{d}\Phi_{\rm II}}{\mathrm{d}x}\right) = v_1 v_2 \epsilon_1(1 - \beta). \qquad [140]$$

There is now enough information to combine [138]–[140] and obtain [133], which is true in either reference frame.

We note that the stagnation pressure, $p_0$, is not unequivocally defined, for the situation represented in figure 7(b), until a statement is made about the choice of reference potential.

### *3.2.3. Energy conservation in macroscopically steady flow, Bernoulli's equation for the dispersed phase*

For the base case the mechanical energy conservation equation is equivalent to Bernoulli's equation and the only useful result to be derived from it is [66].

When the unit cells move at speed $v_2$, the time average of the energy and work flow across a section fixed in space is

$$\langle\dot{E}\rangle = \frac{1}{T}\int[(\tfrac{1}{2}\rho_1 u_1^2 + p)\,u_x\,\mathrm{d}A_1 + (\tfrac{1}{2}\rho_2 v_2^2 - \sigma_x)v_2\,\mathrm{d}A_2]\,\mathrm{d}t. \qquad [141]$$

We use [114] and [115] to transform time integrals to space integrals and [123] with the steady flow version of [127] to simplify the first term on the right-hand side of [141]. The result is

$$\langle\dot{E}\rangle = \frac{1}{L}\int[p_0 + \rho_1 v_2(u_{0x} - \beta\epsilon_1 w)](v_2 + u_{0x})\,\mathrm{d}V_1 + \frac{1}{L}\int(\tfrac{1}{2}\rho_2 v_2^2 - \sigma_x)v_2\,\mathrm{d}V_2. \qquad [142]$$

The term involving $\sigma_x$ in [142] is the average rate at which work is being done by the solid phase as it passes the measurement plane. This may also be evaluated by considering the work done by the part of the interphase boundary which has crossed this plane as the entire particle passes by, namely $\int p\mathbf{x}\cdot\mathbf{ds}'$. Since the rate at which particles pass by is $v_2/L$, the rate at which they transfer work to fluid beyond the measurement plane is $(v_2/L)\int p\mathbf{x}\cdot\mathbf{ds}'$, which is compatible with the final term in [142] in view of [63].

When all the terms are worked out in [142], we obtain, using [4] and [118],

$$\frac{\langle \dot{E} \rangle}{A} = p_0 \epsilon_1 v_1 + v_2 \epsilon_1 \rho_1 \langle u_{0x}^2 \rangle + \epsilon_1 \rho_1 w v_2 (v_2 - \epsilon_1 \beta v_1) + \tfrac{1}{2} \rho_2 \epsilon_2 v_2^3 - v_2 \epsilon_2 \langle \sigma_x \rangle, \qquad [143]$$

which can be rearranged, using [79], to

$$\frac{\langle \dot{E} \rangle}{A} = p_0(\epsilon_1 v_1 + \epsilon_2 v_2) + v_2 \Bigg[ -p_0 \epsilon_2 + \tfrac{1}{2} \rho_2 \epsilon_2 v_2^2 - \rho_1 \epsilon_1 v_2 w(\epsilon_1 \beta - 1)$$

$$+ \epsilon_2 \left( p_1 + \tfrac{1}{2} \rho_1 w^2 \epsilon_1^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) \right) \Bigg]. \qquad [144]$$

The first term on the right in [144] is the product of the stagnation pressure and the overall volumetric flux. If both phases originate in a region, designated by the subscript "s", where each has negligible velocity and a common stagnation pressure, the net energy flow leaving this region is

$$\dot{E}_s = p_0 A_s (\epsilon_1 v_1 + \epsilon_2 v_2)_s. \qquad [145]$$

Moreover, from the continuity equation for each phase

$$Q_1 = A_s(\epsilon_1 v_1)_s = A \epsilon_1 v_1 = A j_1 \qquad [146]$$

and

$$Q_2 = A_s(\epsilon_2 v_2)_s = A \epsilon_2 v_2 = A j_2. \qquad [147]$$

Using [145]–[147] and equating $\dot{E}_s$ with $\dot{E}$, because of energy conservation in the absence of external forces or body force fields (note that a similar development is not possible for three-dimensional cells), we find that the term in square brackets in [144] must be zero, i.e.

$$p_0 = p_1 + \tfrac{1}{2} \rho_1 w^2 \epsilon_1^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) + \tfrac{1}{2} \rho_2 v_2^2 - \frac{\epsilon_1 \rho_1}{\epsilon_2} v_2 w(\epsilon_1 \beta - 1). \qquad [148]$$

In the more general case where the phases do not have a common stagnation pressure, $p_0$ is the stagnation pressure of the fluid alone. Equation [144] can then be expressed as

$$\langle \dot{E} \rangle = p_0(Q_1 + Q_2) + Q_2 \left[ -p_0 + \tfrac{1}{2} \rho_2 v_2^2 - \frac{\epsilon_1}{\epsilon_2} \rho_1 v_2 w(\epsilon_1 \beta - 1) + p_1 + \tfrac{1}{2} \rho_1 w^2 \epsilon_1^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) \right]. \qquad [149]$$

As a result of energy conservation and continuity, the term in square brackets in [149] is seen to be invariant of the motion. It is a special form of *Bernoulli's equation for the dispersed phase*, resembling [51] in Wallis (1990) and reducing to the steady flow version of that result when [122] is valid. In general, however, $p_2$ is derived by considering growth of the particles at constant $L$, as in [98], and the final two terms in [149] are not the same as the bulk pressure in the particles but describe a directional stress along the direction of mean motion, derived from [79].

Setting the square-bracket term in [149] equal to "C", which is to be evaluated somewhere in the flow, [148] is valid in the more general form

$$p_0 + \mathrm{C} = p_1 + \tfrac{1}{2} \rho_1 w^2 \epsilon_1^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) + \tfrac{1}{2} \rho_2 v_2^2 - \frac{\epsilon_1}{\epsilon_2} \rho_1 v_2 w(\epsilon_1 \beta - 1). \qquad [150]$$

Equation [150] may be combined with [130] to obtain a relationship between the velocities of the phases which is valid between any points in the flow field:

$$\tfrac{1}{2} \rho_2 v_2^2 - \tfrac{1}{2} \rho_1 v_1^2 - \tfrac{1}{2} \rho_1 (\epsilon_1 \beta - 1) \left[ v_1^2 - v_2^2 + \frac{2\epsilon_1}{\epsilon_2} (v_2 v_1 - v_2^2) \right] + \tfrac{1}{2} \rho_1 w^2 \epsilon_1^2 \left( \frac{\beta}{\epsilon_1} + \frac{L}{\epsilon_2} \frac{\partial \beta}{\partial L} \right) = \mathrm{C}. \qquad [151]$$

The constant C is zero if the phases have a common stagnation pressure.

If C is known, perhaps from the conditions at one point in the system, and the flow rates are specified in [146] and [147], [151] is the third equation which *makes it possible to solve* for $v_1$, $v_2$ and $\epsilon_1$ (or $\epsilon_2$ or $L$), given the other geometrical parameters. For example, these parameters could be the diameter of suspended spheres and the diameter of a straight pipe in which they are flowing.

$L$, $\beta$, $\epsilon_1$ and $\epsilon_2$ are all functions of each other, once the other dimensions are specified. The way in which the two-phase mixture gets from one region that is macroscopically uniform (i.e. unit cells forming part of a "continuum" can be defined) to another is irrelevant; the properties in the two regions are related by [151]. For example, it is not necessary to solve any differential equations or to specify the shape of the inlet nozzle in order to predict the conditions in a straight tube that is fed by given flow rates of spheres and fluid from a reservoir, as long as the flow is *potential.*

### *3.2.4. Minimization of kinetic energy density*

We will now show that [151], with C = 0, is equivalent to minimization, subject to the constraint of known flow rates, of the kinetic energy per unit volume, which is (Wallis 1989)

$$k = \tfrac{1}{2}\epsilon_1\rho_1 v_1^2 + \tfrac{1}{2}\epsilon_2\rho_2 v_2^2 + \tfrac{1}{2}\epsilon_1\rho_1 w^2(\epsilon_1\beta - 1). \qquad [152]$$

The velocities are related to the (known) fluxes of each phase by [146] and [147], and [152] may be expressed as

$$k = \tfrac{1}{2}\rho_1\frac{j_1^2}{\epsilon_1} + \tfrac{1}{2}\rho_2\frac{j_2^2}{\epsilon_2} + \tfrac{1}{2}\epsilon_1\rho_1\left(\frac{j_1}{\epsilon_1} - \frac{j_2}{\epsilon_2}\right)^2(\epsilon_1\beta - 1). \qquad [153]$$

Differentiating [153] with respect to the length of the cell, $L$, and replacing $\partial\epsilon_1/\partial L$ by $-\partial\epsilon_2/\partial L$ we obtain

$$2\frac{\mathrm{d}k}{\mathrm{d}L} = \left[\rho_1\frac{j_1^2}{\epsilon_1^2} - \rho_2\frac{j_2^2}{\epsilon_2^2} + \rho_1(\epsilon_1\beta - 1)\left(\frac{j_1^2}{\epsilon_1^2} + \frac{2j_1j_2}{\epsilon_2^2} - \frac{2\epsilon_1}{\epsilon_2^3}j_2^2 - \frac{j_2^2}{\epsilon_2^2}\right)\right]\frac{\partial\epsilon_2}{\partial L} + \epsilon_1\rho_1 w^2\left(\epsilon_1\frac{\partial\beta}{\partial L} - \beta\frac{\partial\epsilon_2}{\partial L}\right). \qquad [154]$$

Using [121] and [151], [154] reduces to

$$\frac{\mathrm{d}k}{\mathrm{d}L} = \frac{\epsilon_2}{L}\mathrm{C} \qquad [155]$$

and therefore the kinetic energy per unit volume of the entire flow is a minimum if C = 0.

Interesting conclusions may also be drawn about the *flow force*. We replace the first $p_1$ in [120] using [130] (with $\partial\Phi/\partial t = 0$ for steady flow) and the second $p_1$ in [120] using [148]. The result is

$$\frac{\langle F\rangle}{A} = p_0 + \tfrac{1}{2}\epsilon_1\rho_1 v_1^2 + \tfrac{1}{2}\epsilon_2\rho^2 v_2^2 + \tfrac{1}{2}\epsilon_1\rho_1 w^2(\epsilon_1\beta - 1), \qquad [156]$$

which is truly remarkable when compared with [152] and seems to indicate that the flow force is also minimized when C = 0. If C ≠ 0, $\epsilon_2$C should be added to the right-hand side of [156], because [150] rather than [148] is needed in its derivation. Using [152] we then have

$$\frac{\langle F\rangle}{A} = p_0 + k + \epsilon_2\mathrm{C}. \qquad [157]$$

Therefore, differentiating with respect to $L$ and treating C as a constant:

$$\frac{\;\dfrac{\partial\langle F\rangle}{\partial L}\;}{A} = \frac{\partial k}{\partial L} + \mathrm{C}\frac{\partial\epsilon_2}{\partial L}. \qquad [158]$$

Using [121] and [155] it is found that [158] is identical to

$$\frac{\partial\langle F\rangle}{\partial L} = 0, \qquad [159]$$

which is valid independent of the value of C.

Although [159] appears to show that $\langle F\rangle$ is a minimum, further study may be required. The constraints in all of the above differentiation were that the flow rates of the phases $Q_1$ and $Q_2$ (and hence $j_1$ and $j_2$ in a given pipe) were specified and C was constant. This makes it possible to solve for $v_1$ and $v_2$, using [146], [147] and [151] and one is not at liberty, in general, to vary a further function $\langle F\rangle$.

### *3.2.5. Existence of solutions*

Equation [151] may have multiple solutions or no real solutions, depending on the system properties and the flow rates. The existence of two coincident solutions would indicate that small changes, or a stationary wave, could occur in a duct of constant area transporting given flow rates.

In the special case where $\mathrm{C} = 0$ (common stagnation pressures), [151] is a quadratic for the velocity ratio $(v_2/v_1)$. If the system is an isotropic dispersion, we may use [121] to convert $(L/\epsilon_2)\partial\beta/\partial L$ to $(-\partial\beta/\partial\epsilon_2)$ and make use of the functional dependence of $\beta$ on $\epsilon_2$. For example, if Maxwell's expression for the exertia (Wallis 1989) is chosen:

$$\epsilon_1\beta - 1 = \frac{\epsilon_2}{2} \qquad [160]$$

then, by differentiation,

$$\frac{\beta}{\epsilon_1} - \frac{\partial\beta}{\partial\epsilon_2} = -\frac{1}{2} \qquad [161]$$

and [151] can be solved to yield

$$\frac{v_1}{v_2} = \left(\frac{1 + 2\dfrac{\rho_2}{\rho_1}}{3}\right)^{1/2}, \qquad [162]$$

which was previously derived by Wallis (1989).

If, on the other hand, Guerst's (1985) "marginal stability" expression is used,

$$\epsilon_1\beta - 1 = \frac{\epsilon_2}{2}(1 - 3\epsilon_2), \qquad [163]$$

we have

$$\beta = 1 + \tfrac{3}{2}\epsilon_2 \qquad [164]$$

which is equivalent to *polarization* (Wallis 1991) with a dipole moment that is unaffected by the interactions between particles. The solution to [151] with $\mathrm{C} = 0$ is then

$$\frac{1}{2}\frac{w\epsilon_1}{v_2} = \left(\frac{2 + \dfrac{\rho_2}{\rho_1}}{3}\right)^{1/2} - 1. \qquad [164]$$

To investigate more general conditions for the existence of solutions we rearrange [151] to

$$\tfrac{1}{2}(\rho_2 - \rho_1)v_2^2 - \rho_1 v_2 w\frac{\epsilon_1}{\epsilon_2}(\beta - 1) + \tfrac{1}{2}\rho_1 w^2\epsilon_1^2\frac{L}{\epsilon_2}\frac{\partial\beta}{\partial L} = \mathrm{C}. \qquad [166]$$

With $\mathrm{C} = 0$, [166] is a quadratic for $w\epsilon_1/v_2\epsilon_2$, which has real solutions if

$$\rho_1(\beta - 1)^2 > (\rho_2 - \rho_1)\epsilon_2 L\frac{\partial\beta}{\partial L} \qquad [167]$$

or, for an isotropic dispersion, if

$$\rho_1(\beta - 1)^2 > -(\rho_2 - \rho_1)\epsilon_2^2\frac{\partial\beta}{\partial\epsilon_2}. \qquad [168]$$

Equal roots occur, giving a condition for stationary waves, when [167] is an equality and then

$$\frac{w\epsilon_1}{v_2\epsilon_2} = \frac{\dfrac{\rho_2}{\rho_1} - 1}{\beta - 1} = \frac{\beta - 1}{\epsilon_2 L\dfrac{\partial\beta}{\partial L}}. \qquad [169]$$

The "structural properties", $(\beta - 1)$ and $\partial\beta/\partial L$ occurring in [167] and [169], are the same as those appearing in [74]. We note that a simple criterion for "static stability" of the particle matrix in figure 3 is that $\partial F_e/\partial L$ be positive, i.e.

$$2\frac{\partial\beta}{\partial L} + L\frac{\partial^2\beta}{\partial L^2} < 0 \qquad [170]$$

which becomes, using [121] for an isotropic structure,

$$\frac{\partial^2\beta}{\partial\epsilon_2^2} < 0. \qquad [171]$$

If a general expression for the exertia is chosen of the form

$$\epsilon_1\beta - 1 = \frac{\epsilon_2}{2}(1 - n\epsilon_2), \qquad [172]$$

then [171] will be satisfied if

$$n > 3. \qquad [173]$$

Marginal stability occurs if $n = 3$, which leads to Geurst's equation [164]. In this case $F_e$ is identically zero in [74] and the particle matrix has zero compressibility. Equation [169] gives the same result only if $w\epsilon_1 = -v_2$, i.e. if $v_2 = -j_0$, which corresponds to a system in which the net flux $(\epsilon_1 v_1 + \epsilon_2 v_2)$ is zero, which is unlikely to occur in a flow originating from a common stagnation pressure unless additional external forces play a role.

## 4. STRESSES, MOMENTUM FLUX AND BERNOULLI'S EQUATION FOR THREE-DIMENSIONAL CELLS

### 4.1. Base Case

Consider the brick-like unit cell described earlier, with overall dimensions $X$, $Y$ and $Z$. No solid walls are involved and there is flow through all sides of the cell, the fluxes being related to the corresponding potential differences by [8].

The *fluid kinetic energy* per unit total volume is

$$k = \tfrac{1}{2}\rho_1 \sum j_{0i} A_i (\phi_{B_{0i}} - \phi_{A_{0i}}) \qquad [174]$$

or, using [8],

$$k = \tfrac{1}{2}\rho_1 \sum \beta_i j_{0i}^2 = \tfrac{1}{2}\rho_1 \epsilon_1^2 \sum \beta_i w_i^2. \qquad [175]$$

As a corollary of [175] the average value of $u^2$, over the volume of the fluid, is

$$\langle u_0^2 \rangle = \epsilon_1 \sum \beta_i w_i^2. \qquad [176]$$

Therefore, the equivalent of [66] is *Bernoulli's equation* in the form

$$p_1 + \frac{1}{2}\rho_1\epsilon_1 \sum \beta_i w_i^2 = p_1 + \frac{1}{2}\frac{\rho_1}{\epsilon_1}\sum \beta_i j_{0i}^2 = p_0. \qquad [177]$$

The force function in the $x$-direction is still given by [56], but now the Reynolds stresses result from fluctuations produced by the three components of the relative velocity, $w_i$. We adopt the nomenclature that $u_{0xi}$ is the $x$-component of the microscopic fluid velocity in the base case resulting from the component $w_i$.

From the symmetry of the cell,

$$\langle u_{0ij} \rangle = 0, \quad \langle u_{0xi} u_{0xj} \rangle = 0; \quad i \neq j. \qquad [178]$$

Equation [176] is actually the sum of the three components,

$$\langle u_{0xi}^2 + u_{0yi}^2 + u_{0zi}^2 \rangle = \epsilon_1 \beta_i w_i^2, \qquad [179]$$

and [56] may be expressed as

$$\frac{F_x}{A_x} = \epsilon_1 p_1 + \rho_1 \epsilon_1 \sum \langle u_{0xi}^2 \rangle - \epsilon_2 \langle \sigma_x \rangle. \qquad [180]$$

To obtain the equivalent of [78] and [79] we consider the system sketched in figure 8. It resembles figure 3 except that the portion which contains particles is made up of brick-like cells having dimensions $X$, $Y$ and $Z$. There are superposed potential differences $\Delta\phi_x$, $\Delta\phi_y$ and $\Delta\phi_z$ over the entire system.

The overall fluid kinetic energy can be written in a more general form of [68]:

$$K = \tfrac{1}{2}\rho_1\left(\frac{\Delta\phi_x^2}{B_x} + \frac{\Delta\phi_y^2}{B_y} + \frac{\Delta\phi_z^2}{B_z}\right). \qquad [181]$$

The resistance in the $x$-direction is made up of two parts in series and [67] is again valid, therefore

$$B_x = \frac{l}{YZ}(1 - f + f\beta_x). \qquad [182]$$

In the other directions the resistances of the open cells and those containing particles are in parallel, therefore

$$\frac{1}{B_y} = \frac{lZ}{Y}(1 - f) + \frac{lZ}{Y}\frac{f}{\beta_y} \qquad [183]$$

and

$$\frac{1}{B_z} = \frac{lY}{Z}(1 - f) + \frac{lY}{Z}\frac{f}{\beta_z}. \qquad [184]$$

The potential differences can also be expressed as

$$\Delta\phi_x = YZB_x j_{0x}, \quad \Delta\phi_y = Y\beta_y j_{0y}, \quad \Delta\phi_z = Z\beta_z j_{0z}, \qquad [185]$$

where $j_{0y}$ and $j_{0z}$ are the fluxes in the cells *containing particles* in figure 8.

When [182]–[184] are used in [181] (the result is differentiated with respect to $f$, keeping $\Delta\phi_i$ constant, and [71] is used, with $X$ replacing $L$), we obtain

$$\frac{2}{\rho_1 lYZ}\frac{\mathrm{d}K}{\mathrm{d}f} = j_{0x}^2 - j_{0x}^2\beta_x + j_{0y}^2\beta_y + j_{0z}^2\beta_z - j_{0y}^2\beta_y^2 - j_{0z}^2\beta_z^2 - \sum_i j_{0i}^2 X\frac{\partial\beta_i}{\partial X}. \qquad [186]$$

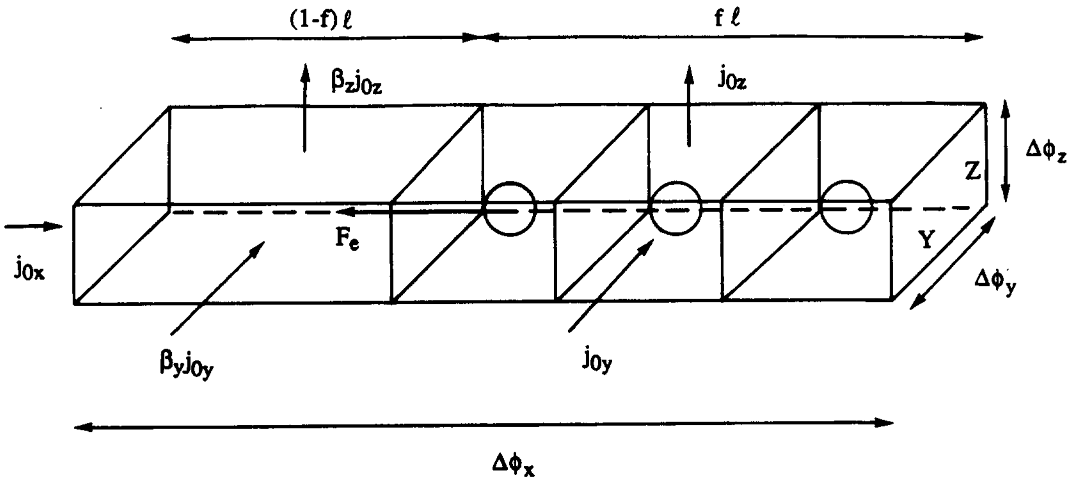

Figure 8. The equivalent of figure 3 for three-dimensional cells.

The energy theorem gives

$$\frac{F_e}{A_x} = \frac{F_e}{YZ} = \frac{1}{lYZ}\frac{dK}{df}. \qquad [187]$$

We now wish to follow a sequence of reasoning analogous to the derivations of [75]–[79]. Instead of [75] we have

$$\frac{F_x}{A_x} = -\frac{F_e}{YZ} + p_a + \rho_1 j_{0x}^2. \qquad [188]$$

In deriving the equivalent of [77] we must take into account the three velocity components in the particle-free region. The $y$-component is equal to $\Delta\phi_y/Y$, which is the same as $j_{0y}\beta_y$ because of [185]. Therefore, [77] becomes

$$p_a = p_0 - \tfrac{1}{2}\rho_1(j_{0x}^2 + \beta_y^2 j_{0y}^2 + \beta_z^2 j_{0z}^2). \qquad [189]$$

Using [186]–[189] and invoking [177], the equivalent of [78] for the *flow force* is

$$\frac{F_x}{A_x} = p_1 + \rho_1\beta_x j_{0x}^2 + \tfrac{1}{2}\rho_1\sum_i j_{0i}^2\left(\frac{\epsilon_2}{\epsilon_1}\beta_i + X\frac{\partial\beta_i}{\partial X}\right). \qquad [190]$$

Combining [180] with [190] and rearranging we obtain

$$\rho_1\epsilon_1\sum_i\langle u_{0xi}^2\rangle - \epsilon_2\langle\sigma_x\rangle = \rho_1\beta_x j_{0x}^2 + \epsilon_2\left[p_1 + \tfrac{1}{2}\rho_1\sum_i j_{0i}^2\left(\frac{\beta_i}{\epsilon_1} + \frac{X}{\epsilon_2}\frac{\partial\beta_i}{\partial X}\right)\right]. \qquad [191]$$

Similar equations are valid in the $y$- and $z$-directions. Adding all three together and using [179] leads to

$$\rho_1\sum_i\beta_i j_{0i}^2 + 3\epsilon_2 p_2 = 3\epsilon_2 p_1 + \tfrac{1}{2}\rho_1\left[\sum_{i,j} j_{0i}^2\left(2\beta_i + 3\frac{\beta_i\epsilon_2}{\epsilon_1} + X_j\frac{\partial\beta_i}{\partial X_j}\right)\right], \qquad [192]$$

which may be reduced to

$$p_2 = p_1 + \frac{\rho_1}{2}\sum_i j_{0i}^2\left(\frac{\beta_i}{\epsilon_1} + \frac{1}{3}\sum_j\frac{X_j}{\epsilon_2}\frac{\partial\beta_i}{\partial X_j}\right) \qquad [193]$$

which gives the *pressure difference between the phases.*

Now, $p_2$ may be computed independently from the energy method by considering a perturbation in which the particle grows, under uniform strain, and changes the fluid kinetic energy in a unit cell while the applied potentials stay constant. The result, from [175], is

$$p_2 = p_0 - \tfrac{1}{2}\rho_1\sum j_{0i}^2\left(\frac{\partial\beta_i}{\partial\epsilon_2}\right)_{X,Y,Z}. \qquad [194]$$

Moreover, for each of the $\beta_i$:

$$d\beta_i = \sum_j\frac{\partial\beta_i}{\partial X_j}dX_j + \frac{\partial\beta_i}{\partial\epsilon_2}d\epsilon_2; \qquad [195]$$

and, as in [109], for a uniform strain of the entire cell

$$\gamma = \frac{dX_j}{X_j} = \frac{1}{3}\frac{d\epsilon_2}{\epsilon_2}, \qquad [196]$$

in which case the $\beta_i$ are unchanged and, using [196] in [195],

$$d\beta_i = 0 = \sum_j X_j\frac{\partial\beta_i}{\partial X_j} + 3\epsilon_2\frac{\partial\beta_i}{\partial\epsilon_2}. \qquad [197]$$

Use of [177], [194] and [197] provides an alternative derivation of [193] and checks its validity.

### *4.2. Moving Cells*

To derive the *averaged Bernoulli equation* we integrate [123] over the cell as before. The results equivalent to [127]–[130] are:

$$\frac{\partial \phi}{\partial t} = \mathbf{v}_2 \cdot (\mathbf{u}_0 - \epsilon_1 \boldsymbol{\beta} \cdot \mathbf{w}) + \frac{\partial \Phi}{\partial t}, \qquad [198]$$

$$p_1 = \frac{1}{V_1} \int \left[ p_0 - \tfrac{1}{2} \rho_1 (\mathbf{v}_2 + \mathbf{u}_0)^2 + \rho_1 \mathbf{v}_2 \cdot (\mathbf{u}_0 - \epsilon_1 \boldsymbol{\beta} \cdot \mathbf{w}) + \rho_1 \frac{\partial \Phi}{\partial t} \right] \mathrm{d}V_1, \qquad [199]$$

$$p_1 = p_0 - \tfrac{1}{2} \rho_1 v_2^2 - \rho_1 \epsilon_1 \sum_j \left( \frac{\beta_j w_j^2}{2} + \beta_j v_{2j} w_j \right) + \rho_1 \frac{\partial \Phi}{\partial t} \qquad [200]$$

and

$$p_1 = p_0 - \tfrac{1}{2} \rho_1 v_1^2 - \tfrac{1}{2} \rho_1 \sum_j (\epsilon_1 \beta_j - 1)(v_{1j}^2 - v_{2j}^2) + \rho_1 \frac{\partial \Phi}{\partial t}. \qquad [201]$$

The flow force follows from the equivalents of [117] and [119]:

$$\left\langle \frac{F}{A} \right\rangle_i = \frac{1}{V} \int \{ [p + \rho_1 (v_{2i} + u_{0i})^2] \, \mathrm{d}V_1 + (\rho_2 v_{2i}^2 - \sigma_i) \, \mathrm{d}V_2 \} \qquad [202]$$

and

$$\left\langle \frac{F}{A} \right\rangle_i = \epsilon_1 p_1 + \epsilon_1 \rho_1 v_{1i}^2 + \epsilon_2 \rho_2 v_{2i}^2 + \epsilon_1 \rho_1 (\langle u_{0i}^2 \rangle - w_i^2) - \epsilon_2 \langle \sigma_i \rangle. \qquad [203]$$

Since

$$u_{0i} = \sum_j u_{0ij}, \qquad [204]$$

we may use [178] to show that

$$\langle u_{0i}^2 \rangle = \sum_j \langle u_{0ij} \rangle^2 \qquad [205]$$

and, therefore, [191] can be combined with [203] with consistent nomenclature to give the *flow force* as

$$\left\langle \frac{F}{A} \right\rangle_i = \epsilon_1 p_1 + \epsilon_1 \rho_1 v_{1i}^2 + \epsilon_2 \rho_2 v_{2i}^2 + \epsilon_1 \rho_1 w_i^2 (\epsilon_1 \beta_i - 1) + \epsilon_2 \left[ p_1 + \tfrac{1}{2} \rho_1 \sum_j j_{0j}^2 \left( \frac{\beta_j}{\epsilon_1} + \frac{X_i}{\epsilon_2} \frac{\partial \beta_j}{\partial X_i} \right) \right] \qquad [206]$$

in place of [120].

In a more general sense, [206] is a normal component of the *combined momentum* flux and stress tensor. For an isotropic dispersion this is (Wallis 1989):

$$\mathbf{P} = \epsilon_1 p_1 \mathbf{I} + \epsilon \rho_1 \mathbf{v}_1 \mathbf{v}_1 + \epsilon_2 \rho_2 \mathbf{v}_2 \mathbf{v}_2 + \epsilon_1 \rho_1 \mathbf{w}\mathbf{w} (\epsilon_1 \beta - 1) + \epsilon_2 p_2 \mathbf{I}, \qquad [207]$$

which makes interesting comparison with [206] and [203]. However, to determine the "shear" components equivalent to [206] involves computation of averages such as $\langle u_{0xi} u_{0yj} \rangle$ and is beyond the scope of this paper. These averages are generally non-zero; e.g. they contribute to the $\epsilon_1^2 \rho_1 w_i w_j \beta$ term in [207].

The average energy flux for steady flow follows from the energy transfer across a plane perpendicular to the $i$-direction due to the passage of a complete unit cell (at velocity $\mathbf{v}_2$). The result corresponding to [142] is

$$\left\langle \frac{\dot{E}}{A} \right\rangle_i = \frac{1}{V} \int \left[ p_0 + \rho_1 \sum_j v_{2j} (u_{0j} - \epsilon_1 \beta_j w_j) \right] (v_{2i} + u_{0i}) \, \mathrm{d}V_1 + \frac{1}{V} \int (\tfrac{1}{2} \rho_2 v_2^2 - \sigma_i) v_{2i} \, \mathrm{d}V_2. \qquad [208]$$

Explicit evaluation of [208] again requires determination of the average $\Sigma \langle u_{0i} u_{0j} \rangle$, which can only be reduced to $\langle u_{0i}^2 \rangle$ when $w_j$ has only one component.

## 5. DISCUSSION

This paper has extended earlier work (Wallis 1989) in which it is shown how the dimensionless resistivity, $\beta$, could be used to determine several properties of a dispersion of particles in a potential fluid flow. Once the geometry of a unit cell is defined, $\beta$ can be determined by solving Laplace's equation. Other properties can then be deduced by straightforward mathematics.

The new features of the present work include:

- Extension to three-dimensional cells in which $\beta$ becomes a tensor.
- A new derivation of the equation of motion of a uniform array, [25] and [26], and the related "two-fluid model" equations, [30]–[33].
- Evaluation of the "interphase impulse" $\int \phi \mathbf{ds}'$ for particles, [15] and [16], and its relation to momentum and kinetic energy density, [41], [44], [50] and [53].
- Evaluation of the "flow force" for stationary, [78] and [190], and moving cells, [120] and [206].
- Derivation of the averaged Bernoulli equation for the fluid for stationary, [66] and [177], and moving cells, [130] and [201].
- Prediction of the mean pressure on the wall of the one-dimensional cells, [89], [107] and [113], and in the particle, [98] and [193].
- Derivation of the energy conservation equation for steady flow with one-dimensional cells, [144], and Bernoulli's equation for phase 2, [149], which makes it possible to solve for local conditions using [151].
- A proof that the kinetic energy density is minimized in one-dimensional flow from a single stagnation pressure.
- Various results concerning the existence of solutions, forces at interfaces, time-varying macroscopic potentials etc.

Unit cells of a simple structure are particularly suitable for the generation of numerical solutions to the microscopic flow field that can be used to check these results and to predict the behavior of specific systems (e.g. spheres in a pipe) (Cai & Wallis 1992). Experimental confirmation is also possible by constructing situations, such as small-amplitude oscillations or rapid accelerations, for which potential flow provides a good approximation (Cai 1992).

It would be interesting to see how these results compare with those involving less orderly, or random, structures that might be analyzed using ensemble averaging. There are also questions about how far a continuum flow can be represented by unit cells that elongate in the direction of extension rather than maintaining an isotropic character.

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